55th International Mathematical Olympiad

PROBLEMS SHORT LIST

WITH SOLUTIONS

IMO 2014

Cape Town - South Africa

Shortlisted Problems with Solutions

55$^{\text{th}}$ International Mathematical Olympiad

Cape Town, South Africa, 2014

Note of Confidentiality

The shortlisted problems should be kept strictly confidential until IMO 2015.

Contributing Countries

The Organising Committee and the Problem Selection Committee of IMO 2014 thank the following 43 countries for contributing 141 problem proposals.

Australia, Austria, Belgium, Benin, Bulgaria, Colombia, Croatia, Cyprus, Czech Republic, Denmark, Ecuador, Estonia, Finland, France, Georgia, Germany, Greece, Hong Kong, Hungary, Iceland, India, Indonesia, Iran, Ireland, Japan, Lithuania, Luxembourg, Malaysia, Mongolia, Netherlands, Nigeria, Pakistan, Russia, Saudi Arabia, Serbia, Slovakia, Slovenia, South Korea, Thailand, Turkey, Ukraine, United Kingdom, U.S.A.

Problem Selection Committee

Johan Meyer
Ilya I. Bogdanov
Géza Kós
Waldemar Pompe
Christian Reiher
Stephan Wagner

Problems

Algebra

A1. Let $z_0 < z_1 < z_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geqslant 1$ such that

$$z_n < \frac{z_0 + z_1 + \cdots + z_n}{n} \leqslant z_{n+1}.$$

(Austria)

A2. Define the function $f\colon (0,1) \to (0,1)$ by

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{if } x < \frac{1}{2}, \\ x^2 & \text{if } x \geqslant \frac{1}{2}. \end{cases}$$

Let a and b be two real numbers such that $0 < a < b < 1$. We define the sequences a_n and b_n by $a_0 = a$, $b_0 = b$, and $a_n = f(a_{n-1})$, $b_n = f(b_{n-1})$ for $n > 0$. Show that there exists a positive integer n such that

$$(a_n - a_{n-1})(b_n - b_{n-1}) < 0.$$

(Denmark)

A3. For a sequence $x_1, x_2, \ldots, x_n$ of real numbers, we define its *price* as

$$\max_{1 \leqslant i \leqslant n} |x_1 + \cdots + x_i|.$$

Given n real numbers, Dave and George want to arrange them into a sequence with a low price. Diligent Dave checks all possible ways and finds the minimum possible price D. Greedy George, on the other hand, chooses x_1 such that $|x_1|$ is as small as possible; among the remaining numbers, he chooses x_2 such that $|x_1 + x_2|$ is as small as possible, and so on. Thus, in the i^{th} step he chooses x_i among the remaining numbers so as to minimise the value of $|x_1 + x_2 + \cdots + x_i|$. In each step, if several numbers provide the same value, George chooses one at random. Finally he gets a sequence with price G.

Find the least possible constant c such that for every positive integer n, for every collection of n real numbers, and for every possible sequence that George might obtain, the resulting values satisfy the inequality $G \leqslant cD$.

(Georgia)

A4. Determine all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f\big(f(m) + n\big) + f(m) = f(n) + f(3m) + 2014$$

for all integers m and n.

(Netherlands)

A5. Consider all polynomials $P(x)$ with real coefficients that have the following property: for any two real numbers x and y one has

$$|y^2 - P(x)| \leqslant 2\,|x| \quad \text{if and only if} \quad |x^2 - P(y)| \leqslant 2\,|y|\,.$$

Determine all possible values of $P(0)$.

(Belgium)

A6. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$n^2 + 4f(n) = f(f(n))^2$$

for all $n \in \mathbb{Z}$.

(United Kingdom)

Combinatorics

C1. Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n+1$ smaller rectangles.

(Serbia)

C2. We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a+b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m.

(Iran)

C3. Let $n \geqslant 2$ be an integer. Consider an $n \times n$ chessboard divided into n^2 unit squares. We call a configuration of n rooks on this board *happy* if every row and every column contains exactly one rook. Find the greatest positive integer k such that for every happy configuration of rooks, we can find a $k \times k$ square without a rook on any of its k^2 unit squares.

(Croatia)

C4. Construct a tetromino by attaching two 2×1 dominoes along their longer sides such that the midpoint of the longer side of one domino is a corner of the other domino. This construction yields two kinds of tetrominoes with opposite orientations. Let us call them S- and Z-tetrominoes, respectively.

S-tetrominoes Z-tetrominoes

Assume that a lattice polygon P can be tiled with S-tetrominoes. Prove than no matter how we tile P using only S- and Z-tetrominoes, we always use an even number of Z-tetrominoes.

(Hungary)

C5. Consider $n \geqslant 3$ lines in the plane such that no two lines are parallel and no three have a common point. These lines divide the plane into polygonal regions; let $\mathcal{F}$ be the set of regions having finite area. Prove that it is possible to colour $\left\lceil\sqrt{n/2}\right\rceil$ of the lines blue in such a way that no region in $\mathcal{F}$ has a completely blue boundary. (For a real number x, $\lceil x \rceil$ denotes the least integer which is not smaller than x.)

(Austria)

C6. We are given an infinite deck of cards, each with a real number on it. For every real number x, there is exactly one card in the deck that has x written on it. Now two players draw disjoint sets A and B of 100 cards each from this deck. We would like to define a rule that declares one of them a winner. This rule should satisfy the following conditions:

1. The winner only depends on the relative order of the 200 cards: if the cards are laid down in increasing order face down and we are told which card belongs to which player, but not what numbers are written on them, we can still decide the winner.
2. If we write the elements of both sets in increasing order as $A = \{a_1, a_2, \ldots, a_{100}\}$ and $B = \{b_1, b_2, \ldots, b_{100}\}$, and $a_i > b_i$ for all i, then A beats B.
3. If three players draw three disjoint sets A, B, C from the deck, A beats B and B beats C, then A also beats C.

How many ways are there to define such a rule? Here, we consider two rules as different if there exist two sets A and B such that A beats B according to one rule, but B beats A according to the other.

(Russia)

C7. Let M be a set of $n \geqslant 4$ points in the plane, no three of which are collinear. Initially these points are connected with n segments so that each point in M is the endpoint of exactly two segments. Then, at each step, one may choose two segments AB and CD sharing a common interior point and replace them by the segments AC and BD if none of them is present at this moment. Prove that it is impossible to perform $n^3/4$ or more such moves.

(Russia)

C8. A card deck consists of 1024 cards. On each card, a set of distinct decimal digits is written in such a way that no two of these sets coincide (thus, one of the cards is empty). Two players alternately take cards from the deck, one card per turn. After the deck is empty, each player checks if he can throw out one of his cards so that each of the ten digits occurs on an even number of his remaining cards. If one player can do this but the other one cannot, the one who can is the winner; otherwise a draw is declared.

Determine all possible first moves of the first player after which he has a winning strategy.

(Russia)

C9. There are n circles drawn on a piece of paper in such a way that any two circles intersect in two points, and no three circles pass through the same point. Turbo the snail slides along the circles in the following fashion. Initially he moves on one of the circles in clockwise direction. Turbo always keeps sliding along the current circle until he reaches an intersection with another circle. Then he continues his journey on this new circle and also changes the direction of moving, i.e. from clockwise to anticlockwise or *vice versa*.

Suppose that Turbo's path entirely covers all circles. Prove that n must be odd.

(India)

Geometry

G1. The points P and Q are chosen on the side BC of an acute-angled triangle ABC so that $\angle PAB = \angle ACB$ and $\angle QAC = \angle CBA$. The points M and N are taken on the rays AP and AQ, respectively, so that $AP = PM$ and $AQ = QN$. Prove that the lines BM and CN intersect on the circumcircle of the triangle ABC.

(Georgia)

G2. Let ABC be a triangle. The points K, L, and M lie on the segments BC, CA, and AB, respectively, such that the lines AK, BL, and CM intersect in a common point. Prove that it is possible to choose two of the triangles ALM, BMK, and CKL whose inradii sum up to at least the inradius of the triangle ABC.

(Estonia)

G3. Let Ω and O be the circumcircle and the circumcentre of an acute-angled triangle ABC with $AB > BC$. The angle bisector of $\angle ABC$ intersects Ω at $M \neq B$. Let Γ be the circle with diameter BM. The angle bisectors of $\angle AOB$ and $\angle BOC$ intersect Γ at points P and Q, respectively. The point R is chosen on the line PQ so that $BR = MR$. Prove that $BR \parallel AC$. (Here we always assume that an angle bisector is a ray.)

(Russia)

G4. Consider a fixed circle Γ with three fixed points A, B, and C on it. Also, let us fix a real number $\lambda \in (0, 1)$. For a variable point $P \notin \{A, B, C\}$ on Γ, let M be the point on the segment CP such that $CM = \lambda \cdot CP$. Let Q be the second point of intersection of the circumcircles of the triangles AMP and BMC. Prove that as P varies, the point Q lies on a fixed circle.

(United Kingdom)

G5. Let $ABCD$ be a convex quadrilateral with $\angle B = \angle D = 90°$. Point H is the foot of the perpendicular from A to BD. The points S and T are chosen on the sides AB and AD, respectively, in such a way that H lies inside triangle SCT and

$$\angle SHC - \angle BSC = 90^\circ, \quad \angle THC - \angle DTC = 90^\circ .$$

Prove that the circumcircle of triangle SHT is tangent to the line BD.

(Iran)

G6. Let ABC be a fixed acute-angled triangle. Consider some points E and F lying on the sides AC and AB, respectively, and let M be the midpoint of EF. Let the perpendicular bisector of EF intersect the line BC at K, and let the perpendicular bisector of MK intersect the lines AC and AB at S and T, respectively. We call the pair (E, F) *interesting*, if the quadrilateral $KSAT$ is cyclic.

Suppose that the pairs (E_1, F_1) and (E_2, F_2) are interesting. Prove that

$$\frac{E_1E_2}{AB} = \frac{F_1F_2}{AC} .$$

(Iran)

G7. Let ABC be a triangle with circumcircle Ω and incentre I. Let the line passing through I and perpendicular to CI intersect the segment BC and the arc BC (not containing A) of Ω at points U and V, respectively. Let the line passing through U and parallel to AI intersect AV at X, and let the line passing through V and parallel to AI intersect AB at Y. Let W and Z be the midpoints of AX and BC, respectively. Prove that if the points I, X, and Y are collinear, then the points I, W, and Z are also collinear.

(U.S.A.)

Number Theory

N1. Let $n \geqslant 2$ be an integer, and let A_n be the set

$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z},\ 0 \leqslant k < n\}.$$

Determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of A_n.

(Serbia)

N2. Determine all pairs (x, y) of positive integers such that

$$\sqrt[3]{7x^2 - 13xy + 7y^2} = |x - y| + 1\,.$$

(U.S.A.)

N3. A coin is called a *Cape Town coin* if its value is $1/n$ for some positive integer n. Given a collection of Cape Town coins of total value at most $99 + \frac{1}{2}$, prove that it is possible to split this collection into at most 100 groups each of total value at most 1.

(Luxembourg)

N4. Let $n > 1$ be a given integer. Prove that infinitely many terms of the sequence $(a_k)_{k \geqslant 1}$, defined by

$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor,$$

are odd. (For a real number x, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.)

(Hong Kong)

N5. Find all triples (p, x, y) consisting of a prime number p and two positive integers x and y such that $x^{p-1} + y$ and $x + y^{p-1}$ are both powers of p.

(Belgium)

N6. Let $a_1 < a_2 < \cdots < a_n$ be pairwise coprime positive integers with a_1 being prime and $a_1 \geqslant n + 2$. On the segment $I = [0, a_1 a_2 \cdots a_n]$ of the real line, mark all integers that are divisible by at least one of the numbers $a_1, \ldots, a_n$. These points split I into a number of smaller segments. Prove that the sum of the squares of the lengths of these segments is divisible by a_1.

(Serbia)

N7. Let $c \geqslant 1$ be an integer. Define a sequence of positive integers by $a_1 = c$ and

$$a_{n+1} = a_n^3 - 4c \cdot a_n^2 + 5c^2 \cdot a_n + c$$

for all $n \geqslant 1$. Prove that for each integer $n \geqslant 2$ there exists a prime number p dividing a_n but none of the numbers $a_1, \ldots, a_{n-1}$.

(Austria)

N8. For every real number x, let $\|x\|$ denote the distance between x and the nearest integer. Prove that for every pair (a, b) of positive integers there exist an odd prime p and a positive integer k satisfying

$$\left\| \frac{a}{p^k} \right\| + \left\| \frac{b}{p^k} \right\| + \left\| \frac{a+b}{p^k} \right\| = 1.$$

(Hungary)

Solutions

Algebra

A1. Let $z_0 < z_1 < z_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geqslant 1$ such that

$$z_n < \frac{z_0 + z_1 + \cdots + z_n}{n} \leqslant z_{n+1}. \tag{1}$$

(Austria)

Solution. For $n = 1, 2, \ldots$ define

$$d_n = (z_0 + z_1 + \cdots + z_n) - n z_n.$$

The sign of d_n indicates whether the first inequality in (1) holds; i.e., it is satisfied if and only if $d_n > 0$.

Notice that

$$n z_{n+1} - (z_0 + z_1 + \cdots + z_n) = (n+1) z_{n+1} - (z_0 + z_1 + \cdots + z_n + z_{n+1}) = -d_{n+1},$$

so the second inequality in (1) is equivalent to $d_{n+1} \leqslant 0$. Therefore, we have to prove that there is a unique index $n \geqslant 1$ that satisfies $d_n > 0 \geqslant d_{n+1}$.

By its definition the sequence $d_1, d_2, \ldots$ consists of integers and we have

$$d_1 = (z_0 + z_1) - 1 \cdot z_1 = z_0 > 0.$$

From

$$d_{n+1} - d_n = \big((z_0 + \cdots + z_n + z_{n+1}) - (n+1) z_{n+1}\big) - \big((z_0 + \cdots + z_n) - n z_n\big) = n(z_n - z_{n+1}) < 0$$

we can see that $d_{n+1} < d_n$ and thus the sequence strictly decreases.

Hence, we have a decreasing sequence $d_1 > d_2 > \ldots$ of integers such that its first element d_1 is positive. The sequence must drop below 0 at some point, and thus there is a unique index n, that is the index of the last positive term, satisfying $d_n > 0 \geqslant d_{n+1}$.

Comment. Omitting the assumption that $z_1, z_2, \ldots$ are integers allows the numbers d_n to be all positive. In such cases the desired n does not exist. This happens for example if $z_n = 2 - \frac{1}{2^n}$ for all integers $n \geqslant 0$.

A2. Define the function $f\colon (0,1) \to (0,1)$ by

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{if } x < \frac{1}{2}, \\ x^2 & \text{if } x \geqslant \frac{1}{2}. \end{cases}$$

Let a and b be two real numbers such that $0 < a < b < 1$. We define the sequences a_n and b_n by $a_0 = a$, $b_0 = b$, and $a_n = f(a_{n-1})$, $b_n = f(b_{n-1})$ for $n > 0$. Show that there exists a positive integer n such that

$$(a_n - a_{n-1})(b_n - b_{n-1}) < 0.$$

(Denmark)

Solution. Note that

$$f(x) - x = \tfrac{1}{2} > 0$$

if $x < \frac{1}{2}$ and

$$f(x) - x = x^2 - x < 0$$

if $x \geqslant \frac{1}{2}$. So if we consider $(0,1)$ as being divided into the two subintervals $I_1 = (0, \frac{1}{2})$ and $I_2 = [\frac{1}{2}, 1)$, the inequality

$$(a_n - a_{n-1})(b_n - b_{n-1}) = \big(f(a_{n-1}) - a_{n-1}\big)\big(f(b_{n-1}) - b_{n-1}\big) < 0$$

holds if and only if a_{n-1} and b_{n-1} lie in distinct subintervals.

Let us now assume, to the contrary, that a_k and b_k always lie in the same subinterval. Consider the distance $d_k = |a_k - b_k|$. If both a_k and b_k lie in I_1, then

$$d_{k+1} = |a_{k+1} - b_{k+1}| = \left|a_k + \tfrac{1}{2} - b_k - \tfrac{1}{2}\right| = d_k.$$

If, on the other hand, a_k and b_k both lie in I_2, then $\min(a_k, b_k) \geqslant \frac{1}{2}$ and $\max(a_k, b_k) = \min(a_k, b_k) + d_k \geqslant \frac{1}{2} + d_k$, which implies

$$d_{k+1} = |a_{k+1} - b_{k+1}| = \left|a_k^2 - b_k^2\right| = \left|(a_k - b_k)(a_k + b_k)\right| \geqslant |a_k - b_k|\left(\tfrac{1}{2} + \tfrac{1}{2} + d_k\right) = d_k(1 + d_k) \geqslant d_k.$$

This means that the difference d_k is non-decreasing, and in particular $d_k \geqslant d_0 > 0$ for all k.

We can even say more. If a_k and b_k lie in I_2, then

$$d_{k+2} \geqslant d_{k+1} \geqslant d_k(1 + d_k) \geqslant d_k(1 + d_0).$$

If a_k and b_k both lie in I_1, then a_{k+1} and b_{k+1} both lie in I_2, and so we have

$$d_{k+2} \geqslant d_{k+1}(1 + d_{k+1}) \geqslant d_{k+1}(1 + d_0) = d_k(1 + d_0).$$

In either case, $d_{k+2} \geqslant d_k(1 + d_0)$, and inductively we get

$$d_{2m} \geqslant d_0(1 + d_0)^m.$$

For sufficiently large m, the right-hand side is greater than 1, but since a_{2m}, b_{2m} both lie in $(0,1)$, we must have $d_{2m} < 1$, a contradiction.

Thus there must be a positive integer n such that a_{n-1} and b_{n-1} do not lie in the same subinterval, which proves the desired statement.

A3. For a sequence $x_1, x_2, \ldots, x_n$ of real numbers, we define its *price* as

$$\max_{1\leqslant i\leqslant n} |x_1 + \cdots + x_i|.$$

Given n real numbers, Dave and George want to arrange them into a sequence with a low price. Diligent Dave checks all possible ways and finds the minimum possible price D. Greedy George, on the other hand, chooses x_1 such that $|x_1|$ is as small as possible; among the remaining numbers, he chooses x_2 such that $|x_1 + x_2|$ is as small as possible, and so on. Thus, in the i^{th} step he chooses x_i among the remaining numbers so as to minimise the value of $|x_1 + x_2 + \cdots + x_i|$. In each step, if several numbers provide the same value, George chooses one at random. Finally he gets a sequence with price G.

Find the least possible constant c such that for every positive integer n, for every collection of n real numbers, and for every possible sequence that George might obtain, the resulting values satisfy the inequality $G \leqslant cD$.

(Georgia)

Answer. $c = 2$.

Solution. If the initial numbers are 1, -1, 2, and -2, then Dave may arrange them as $1, -2, 2, -1$, while George may get the sequence $1, -1, 2, -2$, resulting in $D = 1$ and $G = 2$. So we obtain $c \geqslant 2$.

Therefore, it remains to prove that $G \leqslant 2D$. Let $x_1, x_2, \ldots, x_n$ be the numbers Dave and George have at their disposal. Assume that Dave and George arrange them into sequences $d_1, d_2, \ldots, d_n$ and $g_1, g_2, \ldots, g_n$, respectively. Put

$$M = \max_{1\leqslant i\leqslant n} |x_i|, \quad S = |x_1 + \cdots + x_n|, \quad \text{and} \quad N = \max\{M, S\}.$$

We claim that

$$D \geqslant S, \tag{1}$$

$$D \geqslant \frac{M}{2}, \qquad \text{and} \tag{2}$$

$$G \leqslant N = \max\{M, S\}. \tag{3}$$

These inequalities yield the desired estimate, as $G \leqslant \max\{M, S\} \leqslant \max\{M, 2S\} \leqslant 2D$.

The inequality (1) is a direct consequence of the definition of the price.

To prove (2), consider an index i with $|d_i| = M$. Then we have

$$M = |d_i| = \big|(d_1 + \cdots + d_i) - (d_1 + \cdots + d_{i-1})\big| \leqslant |d_1 + \cdots + d_i| + |d_1 + \cdots + d_{i-1}| \leqslant 2D,$$

as required.

It remains to establish (3). Put $h_i = g_1 + g_2 + \cdots + g_i$. We will prove by induction on i that $|h_i| \leqslant N$. The base case $i = 1$ holds, since $|h_1| = |g_1| \leqslant M \leqslant N$. Notice also that $|h_n| = S \leqslant N$.

For the induction step, assume that $|h_{i-1}| \leqslant N$. We distinguish two cases.

Case 1. Assume that no two of the numbers g_i, g_{i+1}, $\ldots$, g_n have opposite signs.

Without loss of generality, we may assume that they are all nonnegative. Then one has $h_{i-1} \leqslant h_i \leqslant \cdots \leqslant h_n$, thus

$$|h_i| \leqslant \max\big\{|h_{i-1}|, |h_n|\big\} \leqslant N.$$

Case 2. Among the numbers $g_i, g_{i+1}, \ldots, g_n$ there are positive and negative ones.

Then there exists some index $j \geqslant i$ such that $h_{i-1}g_j \leqslant 0$. By the definition of George's sequence we have

$$|h_i| = |h_{i-1} + g_i| \leqslant |h_{i-1} + g_j| \leqslant \max\bigl\{|h_{i-1}|, |g_j|\bigr\} \leqslant N.$$

Thus, the induction step is established.

Comment 1. One can establish the weaker inequalities $D \geqslant \frac{M}{2}$ and $G \leqslant D + \frac{M}{2}$ from which the result also follows.

Comment 2. One may ask a more specific question to find the maximal suitable c if the number n is fixed. For $n = 1$ or 2, the answer is $c = 1$. For $n = 3$, the answer is $c = \frac{3}{2}$, and it is reached e.g., for the collection $1, 2, -4$. Finally, for $n \geqslant 4$ the answer is $c = 2$. In this case the arguments from the solution above apply, and the answer is reached e.g., for the same collection $1, -1, 2, -2$, augmented by several zeroes.

A4. Determine all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f\big(f(m)+n\big)+f(m)=f(n)+f(3m)+2014 \tag{1}$$

for all integers m and n.

(Netherlands)

Answer. There is only one such function, namely $n \longmapsto 2n+1007$.

Solution. Let f be a function satisfying (1). Set $C=1007$ and define the function $g\colon \mathbb{Z} \to \mathbb{Z}$ by $g(m)=f(3m)-f(m)+2C$ for all $m \in \mathbb{Z}$; in particular, $g(0)=2C$. Now (1) rewrites as

$$f\big(f(m)+n\big)=g(m)+f(n)$$

for all $m, n \in \mathbb{Z}$. By induction in both directions it follows that

$$f\big(tf(m)+n\big)=tg(m)+f(n) \tag{2}$$

holds for all $m, n, t \in \mathbb{Z}$. Applying this, for any $r \in \mathbb{Z}$, to the triples $\big(r, 0, f(0)\big)$ and $\big(0, 0, f(r)\big)$ in place of (m, n, t) we obtain

$$f(0)g(r)=f\big(f(r)f(0)\big)-f(0)=f(r)g(0)\,.$$

Now if $f(0)$ vanished, then $g(0)=2C>0$ would entail that f vanishes identically, contrary to (1). Thus $f(0) \neq 0$ and the previous equation yields $g(r)=\alpha\, f(r)$, where $\alpha=\frac{g(0)}{f(0)}$ is some nonzero constant.

So the definition of g reveals $f(3m)=(1+\alpha)f(m)-2C$, i.e.,

$$f(3m)-\beta=(1+\alpha)\big(f(m)-\beta\big) \tag{3}$$

for all $m \in \mathbb{Z}$, where $\beta=\frac{2C}{\alpha}$. By induction on k this implies

$$f(3^k m)-\beta=(1+\alpha)^k\big(f(m)-\beta\big) \tag{4}$$

for all integers $k \geqslant 0$ and m.

Since $3 \nmid 2014$, there exists by (1) some value $d=f(a)$ attained by f that is not divisible by 3. Now by (2) we have $f(n+td)=f(n)+tg(a)=f(n)+\alpha \cdot tf(a)$, i.e.,

$$f(n+td)=f(n)+\alpha \cdot td \tag{5}$$

for all $n, t \in \mathbb{Z}$.

Let us fix any positive integer k with $d \mid (3^k-1)$, which is possible, since $\gcd(3, d)=1$. E.g., by the Euler–Fermat theorem, we may take $k=\varphi(|d|)$. Now for each $m \in \mathbb{Z}$ we get

$$f(3^k m)=f(m)+\alpha(3^k-1)m$$

from (5), which in view of (4) yields $\big((1+\alpha)^k-1\big)\big(f(m)-\beta\big)=\alpha(3^k-1)m$. Since $\alpha \neq 0$, the right hand side does not vanish for $m \neq 0$, wherefore the first factor on the left hand side cannot vanish either. It follows that

$$f(m)=\frac{\alpha(3^k-1)}{(1+\alpha)^k-1} \cdot m+\beta\,.$$

So f is a linear function, say $f(m) = Am + \beta$ for all $m \in \mathbb{Z}$ with some constant $A \in \mathbb{Q}$. Plugging this into (1) one obtains $(A^2 - 2A)m + (A\beta - 2C) = 0$ for all m, which is equivalent to the conjunction of

$$A^2 = 2A \qquad \text{and} \qquad A\beta = 2C\,. \tag{6}$$

The first equation is equivalent to $A \in \{0, 2\}$, and as $C \neq 0$ the second one gives

$$A = 2 \qquad \text{and} \qquad \beta = C\,. \tag{7}$$

This shows that f is indeed the function mentioned in the answer and as the numbers found in (7) do indeed satisfy the equations (6) this function is indeed as desired.

Comment 1. One may see that $\alpha = 2$. A more pedestrian version of the above solution starts with a direct proof of this fact, that can be obtained by substituting some special values into (1), e.g., as follows.

Set $D = f(0)$. Plugging $m = 0$ into (1) and simplifying, we get

$$f(n + D) = f(n) + 2C \tag{8}$$

for all $n \in \mathbb{Z}$. In particular, for $n = 0, D, 2D$ we obtain $f(D) = 2C + D$, $f(2D) = f(D) + 2C = 4C + D$, and $f(3D) = f(2D) + 2C = 6C + D$. So substituting $m = D$ and $n = r - D$ into (1) and applying (8) with $n = r - D$ afterwards we learn

$$f(r + 2C) + 2C + D = \big(f(r) - 2C\big) + (6C + D) + 2C\,,$$

i.e., $f(r + 2C) = f(r) + 4C$. By induction in both directions it follows that

$$f(n + 2Ct) = f(n) + 4Ct \tag{9}$$

holds for all $n, t \in \mathbb{Z}$.

Claim. If a and b denote two integers with the property that $f(n + a) = f(n) + b$ holds for all $n \in \mathbb{Z}$, then $b = 2a$.

Proof. Applying induction in both directions to the assumption we get $f(n + ta) = f(n) + tb$ for all $n, t \in \mathbb{Z}$. Plugging $(n, t) = (0, 2C)$ into this equation and $(n, t) = (0, a)$ into (9) we get $f(2aC) - f(0) = 2bC = 4aC$, and, as $C \neq 0$, the claim follows. $\square$

Now by (1), for any $m \in \mathbb{Z}$, the numbers $a = f(m)$ and $b = f(3m) - f(m) + 2C$ have the property mentioned in the claim, whence we have

$$f(3m) - C = 3\big(f(m) - C\big)\,.$$

In view of (3) this tells us indeed that $\alpha = 2$.

Now the solution may be completed as above, but due to our knowledge of $\alpha = 2$ we get the desired formula $f(m) = 2m + C$ directly without having the need to go through all linear functions. Now it just remains to check that this function does indeed satisfy (1).

Comment 2. It is natural to wonder what happens if one replaces the number 2014 appearing in the statement of the problem by some arbitrary integer B.

If B is odd, there is no such function, as can be seen by using the same ideas as in the above solution.

If $B \neq 0$ is even, however, then the only such function is given by $n \longmapsto 2n + B/2$. In case $3 \nmid B$ this was essentially proved above, but for the general case one more idea seems to be necessary. Writing $B = 3^\nu \cdot k$ with some integers ν and k such that $3 \nmid k$ one can obtain $f(n) = 2n + B/2$ for all n that are divisible by 3^ν in the same manner as usual; then one may use the formula $f(3n) = 3f(n) - B$ to establish the remaining cases.

Finally, in case $B = 0$ there are more solutions than just the function $n \longmapsto 2n$. It can be shown that all these other functions are periodic; to mention just one kind of example, for any even integers r and s the function

$$f(n) = \begin{cases} r & \text{if } n \text{ is even,} \\ s & \text{if } n \text{ is odd,} \end{cases}$$

also has the property under discussion.

A5. Consider all polynomials $P(x)$ with real coefficients that have the following property: for any two real numbers x and y one has

$$|y^2 - P(x)| \leqslant 2\,|x| \quad \text{if and only if} \quad |x^2 - P(y)| \leqslant 2\,|y|\,. \tag{1}$$

Determine all possible values of $P(0)$.

(Belgium)

Answer. The set of possible values of $P(0)$ is $(-\infty, 0) \cup \{1\}$.

Solution.

Part I. We begin by verifying that these numbers are indeed possible values of $P(0)$. To see that each negative real number $-C$ can be $P(0)$, it suffices to check that for every $C > 0$ the polynomial $P(x) = -\left(\frac{2x^2}{C} + C\right)$ has the property described in the statement of the problem. Due to symmetry it is enough for this purpose to prove $|y^2 - P(x)| > 2\,|x|$ for any two real numbers x and y. In fact we have

$$|y^2 - P(x)| = y^2 + \frac{x^2}{C} + \frac{(|x| - C)^2}{C} + 2\,|x| \geqslant \frac{x^2}{C} + 2\,|x| \geqslant 2\,|x|\,,$$

where in the first estimate equality can only hold if $|x| = C$, whilst in the second one it can only hold if $x = 0$. As these two conditions cannot be met at the same time, we have indeed $|y^2 - P(x)| > 2\,|x|$.

To show that $P(0) = 1$ is possible as well, we verify that the polynomial $P(x) = x^2 + 1$ satisfies (1). Notice that for all real numbers x and y we have

$$\begin{aligned}
|y^2 - P(x)| \leqslant 2\,|x| &\iff (y^2 - x^2 - 1)^2 \leqslant 4x^2 \\
&\iff 0 \leqslant \big((y^2 - (x-1)^2\big)\big((x+1)^2 - y^2\big) \\
&\iff 0 \leqslant (y - x + 1)(y + x - 1)(x + 1 - y)(x + 1 + y) \\
&\iff 0 \leqslant \big((x+y)^2 - 1\big)\,\big(1 - (x-y)^2\big)\ .
\end{aligned}$$

Since this inequality is symmetric in x and y, we are done.

Part II. Now we show that no values other than those mentioned in the answer are possible for $P(0)$. To reach this we let P denote any polynomial satisfying (1) and $P(0) \geqslant 0$; as we shall see, this implies $P(x) = x^2 + 1$ for all real x, which is actually more than what we want.

First step: We prove that P is even.

By (1) we have

$$|y^2 - P(x)| \leqslant 2\,|x| \iff |x^2 - P(y)| \leqslant 2\,|y| \iff |y^2 - P(-x)| \leqslant 2\,|x|$$

for all real numbers x and y. Considering just the equivalence of the first and third statement and taking into account that y^2 may vary through $\mathbb{R}_{\geqslant 0}$ we infer that

$$\big[P(x) - 2\,|x|, P(x) + 2\,|x|\big] \cap \mathbb{R}_{\geqslant 0} = \big[P(-x) - 2\,|x|, P(-x) + 2\,|x|\big] \cap \mathbb{R}_{\geqslant 0}$$

holds for all $x \in \mathbb{R}$. We claim that there are infinitely many real numbers x such that $P(x) + 2\,|x| \geqslant 0$. This holds in fact for any real polynomial with $P(0) \geqslant 0$; in order to see this, we may assume that the coefficient of P appearing in front of x is nonnegative. In this case the desired inequality holds for all sufficiently small positive real numbers.

For such numbers x satisfying $P(x) + 2\,|x| \geqslant 0$ we have $P(x) + 2\,|x| = P(-x) + 2\,|x|$ by the previous displayed formula, and hence also $P(x) = P(-x)$. Consequently the polynomial $P(x) - P(-x)$ has infinitely many zeros, wherefore it has to vanish identically. Thus P is indeed even.

Second step: We prove that $P(t) > 0$ for all $t \in \mathbb{R}$.

Let us assume for a moment that there exists a real number $t \neq 0$ with $P(t) = 0$. Then there is some open interval I around t such that $|P(y)| \leqslant 2|y|$ holds for all $y \in I$. Plugging $x = 0$ into (1) we learn that $y^2 = P(0)$ holds for all $y \in I$, which is clearly absurd. We have thus shown $P(t) \neq 0$ for all $t \neq 0$.

In combination with $P(0) \geqslant 0$ this informs us that our claim could only fail if $P(0) = 0$. In this case there is by our first step a polynomial $Q(x)$ such that $P(x) = x^2 Q(x)$. Applying (1) to $x = 0$ and an arbitrary $y \neq 0$ we get $|y\, Q(y)| > 2$, which is surely false when y is sufficiently small.

Third step: We prove that P is a quadratic polynomial.

Notice that P cannot be constant, for otherwise if $x = \sqrt{P(0)}$ and y is sufficiently large, the first part of (1) is false whilst the second part is true. So the degree n of P has to be at least 1. By our first step n has to be even as well, whence in particular $n \geqslant 2$.

Now assume that $n \geqslant 4$. Plugging $y = \sqrt{P(x)}$ into (1) we get $\left|x^2 - P\big(\sqrt{P(x)}\big)\right| \leqslant 2\sqrt{P(x)}$ and hence

$$P\big(\sqrt{P(x)}\big) \leqslant x^2 + 2\sqrt{P(x)}$$

for all real x. Choose positive real numbers x_0, a, and b such that if $x \in (x_0, \infty)$, then $ax^n < P(x) < bx^n$; this is indeed possible, for if $d > 0$ denotes the leading coefficient of P, then $\lim\limits_{x\to\infty} \frac{P(x)}{x^n} = d$, whence for instance the numbers $a = \frac{d}{2}$ and $b = 2d$ work provided that x_0 is chosen large enough.

Now for all sufficiently large real numbers x we have

$$a^{n/2+1}x^{n^2/2} < aP(x)^{n/2} < P\big(\sqrt{P(x)}\big) \leqslant x^2 + 2\sqrt{P(x)} < x^{n/2} + 2b^{1/2}x^{n/2}\,,$$

i.e.

$$x^{(n^2-n)/2} < \frac{1 + 2b^{1/2}}{a^{n/2+1}}\,,$$

which is surely absurd. Thus P is indeed a quadratic polynomial.

Fourth step: We prove that $P(x) = x^2 + 1$.

In the light of our first three steps there are two real numbers $a > 0$ and b such that $P(x) = ax^2 + b$. Now if x is large enough and $y = \sqrt{a}\,x$, the left part of (1) holds and the right part reads $|(1 - a^2)x^2 - b| \leqslant 2\sqrt{a}\,x$. In view of the fact that $a > 0$ this is only possible if $a = 1$. Finally, substituting $y = x + 1$ with $x > 0$ into (1) we get

$$|2x + 1 - b| \leqslant 2x \iff |2x + 1 + b| \leqslant 2x + 2\,,$$

i.e.,

$$b \in [1, 4x + 1] \iff b \in [-4x - 3, 1]$$

for all $x > 0$. Choosing x large enough, we can achieve that at least one of these two statements holds; then both hold, which is only possible if $b = 1$, as desired.

Comment 1. There are some issues with this problem in that its most natural solutions seem to use some basic facts from analysis, such as the continuity of polynomials or the intermediate value theorem. Yet these facts are intuitively obvious and implicitly clear to the students competing at this level of difficulty, so that the Problem Selection Committee still thinks that the problem is suitable for the IMO.

Comment 2. It seems that most solutions will in the main case, where $P(0)$ is nonnegative, contain an argument that is somewhat asymptotic in nature showing that P is quadratic, and some part narrowing that case down to $P(x) = x^2 + 1$.

Comment 3. It is also possible to skip the first step and start with the second step directly, but then one has to work a bit harder to rule out the case $P(0) = 0$. Let us sketch one possibility of doing this: Take the auxiliary polynomial $Q(x)$ such that $P(x) = xQ(x)$. Applying (1) to $x = 0$ and an arbitrary $y \neq 0$ we get $|Q(y)| > 2$. Hence we either have $Q(z) \geqslant 2$ for all real z or $Q(z) \leqslant -2$ for all real z. In particular there is some $\eta \in \{-1, +1\}$ such that $P(\eta) \geqslant 2$ and $P(-\eta) \leqslant -2$. Substituting $x = \pm\eta$ into (1) we learn

$$|y^2 - P(\eta)| \leqslant 2 \iff |1 - P(y)| \leqslant 2\,|y| \iff |y^2 - P(-\eta)| \leqslant 2\,.$$

But for $y = \sqrt{P(\eta)}$ the first statement is true, whilst the third one is false.

Also, if one has not obtained the evenness of P before embarking on the fourth step, one needs to work a bit harder there, but not in a way that is likely to cause major difficulties.

Comment 4. Truly curious people may wonder about the set of all polynomials having property (1). As explained in the solution above, $P(x) = x^2 + 1$ is the only one with $P(0) = 1$. On the other hand, it is not hard to notice that for negative $P(0)$ there are more possibilities than those mentioned above. E.g., as remarked by the proposer, if a and b denote two positive real numbers with $ab > 1$ and Q denotes a polynomial attaining nonnegative values only, then $P(x) = -\big(ax^2 + b + Q(x)\big)$ works.

More generally, it may be proved that if $P(x)$ satisfies (1) and $P(0) < 0$, then $-P(x) > 2\,|x|$ holds for all $x \in \mathbb{R}$ so that one just considers the equivalence of two false statements. One may generate all such polynomials P by going through all combinations of a solution of the polynomial equation

$$x = A(x)B(x) + C(x)D(x)$$

and a real $E > 0$, and setting

$$P(x) = -\big(A(x)^2 + B(x)^2 + C(x)^2 + D(x)^2 + E\big)$$

for each of them.

A6. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$n^2 + 4f(n) = f(f(n))^2 \tag{1}$$

for all $n \in \mathbb{Z}$.

(United Kingdom)

Answer. The possibilities are:

- $f(n) = n + 1$ for all n;
- or, for some $a \geqslant 1$, $f(n) = \begin{cases} n+1, & n > -a, \\ -n+1, & n \leqslant -a; \end{cases}$
- or $f(n) = \begin{cases} n+1, & n > 0, \\ 0, & n = 0, \\ -n+1, & n < 0. \end{cases}$

Solution 1.

Part I. Let us first check that each of the functions above really satisfies the given functional equation. If $f(n) = n + 1$ for all n, then we have

$$n^2 + 4f(n) = n^2 + 4n + 4 = (n + 2)^2 = f(n + 1)^2 = f(f(n))^2.$$

If $f(n) = n + 1$ for $n > -a$ and $f(n) = -n + 1$ otherwise, then we have the same identity for $n > -a$ and

$$n^2 + 4f(n) = n^2 - 4n + 4 = (2 - n)^2 = f(1 - n)^2 = f(f(n))^2$$

otherwise. The same applies to the third solution (with $a = 0$), where in addition one has

$$0^2 + 4f(0) = 0 = f(f(0))^2.$$

Part II. It remains to prove that these are really the only functions that satisfy our functional equation. We do so in three steps:

Step 1: We prove that $f(n) = n + 1$ for $n > 0$.

Consider the sequence (a_k) given by $a_k = f^k(1)$ for $k \geqslant 0$. Setting $n = a_k$ in (1), we get

$$a_k^2 + 4a_{k+1} = a_{k+2}^2.$$

Of course, $a_0 = 1$ by definition. Since $a_2^2 = 1 + 4a_1$ is odd, a_2 has to be odd as well, so we set $a_2 = 2r + 1$ for some $r \in \mathbb{Z}$. Then $a_1 = r^2 + r$ and consequently

$$a_3^2 = a_1^2 + 4a_2 = (r^2 + r)^2 + 8r + 4.$$

Since $8r + 4 \neq 0$, $a_3^2 \neq (r^2 + r)^2$, so the difference between a_3^2 and $(r^2 + r)^2$ is at least the distance from $(r^2 + r)^2$ to the nearest even square (since $8r + 4$ and $r^2 + r$ are both even). This implies that

$$|8r + 4| = \left|a_3^2 - (r^2 + r)^2\right| \geqslant (r^2 + r)^2 - (r^2 + r - 2)^2 = 4(r^2 + r - 1),$$

(for $r = 0$ and $r = -1$, the estimate is trivial, but this does not matter). Therefore, we ave

$$4r^2 \leqslant |8r + 4| - 4r + 4.$$

If $|r| \geqslant 4$, then

$$4r^2 \geqslant 16|r| \geqslant 12|r| + 16 > 8|r| + 4 + 4|r| + 4 \geqslant |8r + 4| - 4r + 4,$$

a contradiction. Thus $|r| < 4$. Checking all possible remaining values of r, we find that $(r^2 + r)^2 + 8r + 4$ is only a square in three cases: $r = -3$, $r = 0$ and $r = 1$. Let us now distinguish these three cases:

- $r = -3$, thus $a_1 = 6$ and $a_2 = -5$. For each $k \geqslant 1$, we have

  $$a_{k+2} = \pm\sqrt{a_k^2 + 4a_{k+1}},$$

  and the sign needs to be chosen in such a way that $a_{k+1}^2 + 4a_{k+2}$ is again a square. This yields $a_3 = -4$, $a_4 = -3$, $a_5 = -2$, $a_6 = -1$, $a_7 = 0$, $a_8 = 1$, $a_9 = 2$. At this point we have reached a contradiction, since $f(1) = f(a_0) = a_1 = 6$ and at the same time $f(1) = f(a_8) = a_9 = 2$.

- $r = 0$, thus $a_1 = 0$ and $a_2 = 1$. Then $a_3^2 = a_1^2 + 4a_2 = 4$, so $a_3 = \pm 2$. This, however, is a contradiction again, since it gives us $f(1) = f(a_0) = a_1 = 0$ and at the same time $f(1) = f(a_2) = a_3 = \pm 2$.

- $r = 1$, thus $a_1 = 2$ and $a_2 = 3$. We prove by induction that $a_k = k + 1$ for all $k \geqslant 0$ in this case, which we already know for $k \leqslant 2$ now. For the induction step, assume that $a_{k-1} = k$ and $a_k = k + 1$. Then

  $$a_{k+1}^2 = a_{k-1}^2 + 4a_k = k^2 + 4k + 4 = (k + 2)^2,$$

  so $a_{k+1} = \pm(k + 2)$. If $a_{k+1} = -(k + 2)$, then

  $$a_{k+2}^2 = a_k^2 + 4a_{k+1} = (k + 1)^2 - 4k - 8 = k^2 - 2k - 7 = (k - 1)^2 - 8.$$

  The latter can only be a square if $k = 4$ (since 1 and 9 are the only two squares whose difference is 8). Then, however, $a_4 = 5$, $a_5 = -6$ and $a_6 = \pm 1$, so

  $$a_7^2 = a_5^2 + 4a_6 = 36 \pm 4,$$

  but neither 32 nor 40 is a perfect square. Thus $a_{k+1} = k + 2$, which completes our induction. This also means that $f(n) = f(a_{n-1}) = a_n = n + 1$ for all $n \geqslant 1$.

Step 2: We prove that either $f(0) = 1$, or $f(0) = 0$ and $f(n) \neq 0$ for $n \neq 0$.

Set $n = 0$ in (1) to get

$$4f(0) = f(f(0))^2.$$

This means that $f(0) \geqslant 0$. If $f(0) = 0$, then $f(n) \neq 0$ for all $n \neq 0$, since we would otherwise have

$$n^2 = n^2 + 4f(n) = f(f(n))^2 = f(0)^2 = 0.$$

If $f(0) > 0$, then we know that $f(f(0)) = f(0) + 1$ from the first step, so

$$4f(0) = \big(f(0) + 1\big)^2,$$

which yields $f(0) = 1$.

Step 3: We discuss the values of $f(n)$ for $n < 0$.

Lemma. For every $n \geqslant 1$, we have $f(-n) = -n+1$ or $f(-n) = n+1$. Moreover, if $f(-n) = -n+1$ for some $n \geqslant 1$, then also $f(-n+1) = -n+2$.

Proof. We prove this statement by strong induction on n. For $n = 1$, we get

$$1 + 4f(-1) = f(f(-1))^2.$$

Thus $f(-1)$ needs to be nonnegative. If $f(-1) = 0$, then $f(f(-1)) = f(0) = \pm 1$, so $f(0) = 1$ (by our second step). Otherwise, we know that $f(f(-1)) = f(-1) + 1$, so

$$1 + 4f(-1) = \big(f(-1) + 1\big)^2,$$

which yields $f(-1) = 2$ and thus establishes the base case. For the induction step, we consider two cases:

- If $f(-n) \leqslant -n$, then

$$f(f(-n))^2 = (-n)^2 + 4f(-n) \leqslant n^2 - 4n < (n-2)^2,$$

  so $|f(f(-n))| \leqslant n-3$ (for $n = 2$, this case cannot even occur). If $f(f(-n)) \geqslant 0$, then we already know from the first two steps that $f(f(f(-n))) = f(f(-n)) + 1$, unless perhaps if $f(0) = 0$ and $f(f(-n)) = 0$. However, the latter would imply $f(-n) = 0$ (as shown in Step 2) and thus $n = 0$, which is impossible. If $f(f(-n)) < 0$, we can apply the induction hypothesis to $f(f(-n))$. In either case, $f(f(f(-n))) = \pm f(f(-n)) + 1$. Therefore,

$$f(-n)^2 + 4f(f(-n)) = f(f(f(-n)))^2 = \big(\pm f(f(-n)) + 1\big)^2,$$

  which gives us

$$\begin{aligned} n^2 \leqslant f(-n)^2 &= \big(\pm f(f(-n)) + 1\big)^2 - 4f(f(-n)) \leqslant f(f(-n))^2 + 6|f(f(-n))| + 1 \\ &\leqslant (n-3)^2 + 6(n-3) + 1 = n^2 - 8, \end{aligned}$$

  a contradiction.

- Thus, we are left with the case that $f(-n) > -n$. Now we argue as in the previous case: if $f(-n) \geqslant 0$, then $f(f(-n)) = f(-n) + 1$ by the first two steps, since $f(0) = 0$ and $f(-n) = 0$ would imply $n = 0$ (as seen in Step 2) and is thus impossible. If $f(-n) < 0$, we can apply the induction hypothesis, so in any case we can infer that $f(f(-n)) = \pm f(-n) + 1$. We obtain

$$(-n)^2 + 4f(-n) = \big(\pm f(-n) + 1\big)^2,$$

  so either

$$n^2 = f(-n)^2 - 2f(-n) + 1 = \big(f(-n) - 1\big)^2,$$

  which gives us $f(-n) = \pm n + 1$, or

$$n^2 = f(-n)^2 - 6f(-n) + 1 = \big(f(-n) - 3\big)^2 - 8.$$

  Since 1 and 9 are the only perfect squares whose difference is 8, we must have $n = 1$, which we have already considered.

Finally, suppose that $f(-n) = -n+1$ for some $n \geqslant 2$. Then

$$f(-n+1)^2 = f(f(-n))^2 = (-n)^2 + 4f(-n) = (n-2)^2,$$

so $f(-n+1) = \pm(n-2)$. However, we already know that $f(-n+1) = -n+2$ or $f(-n+1) = n$, so $f(-n+1) = -n+2$. $\square$

Combining everything we know, we find the solutions as stated in the answer:

- One solution is given by $f(n) = n + 1$ for all n.
- If $f(n)$ is not always equal to $n + 1$, then there is a largest integer m (which cannot be positive) for which this is not the case. In view of the lemma that we proved, we must then have $f(n) = -n+1$ for any integer $n < m$. If $m = -a < 0$, we obtain $f(n) = -n+1$ for $n \leqslant -a$ (and $f(n) = n + 1$ otherwise). If $m = 0$, we have the additional possibility that $f(0) = 0$, $f(n) = -n + 1$ for negative n and $f(n) = n + 1$ for positive n.

Solution 2. Let us provide an alternative proof for Part II, which also proceeds in several steps.

Step 1. Let a be an arbitrary integer and $b = f(a)$. We first concentrate on the case where $|a|$ is sufficiently large.

1. If $b = 0$, then (1) applied to a yields $a^2 = f(f(a))^2$, thus
$$f(a) = 0 \quad \Rightarrow \quad a = \pm f(0). \tag{2}$$
From now on, we set $D = |f(0)|$. Throughout Step 1, we will assume that $a \notin \{-D, 0, D\}$, thus $b \neq 0$.

2. From (1), noticing that $f(f(a))$ and a have the same parity, we get
$$0 \neq 4|b| = \left|f(f(a))^2 - a^2\right| \geqslant a^2 - \big(|a| - 2\big)^2 = 4|a| - 4.$$
Hence we have
$$|b| = |f(a)| \geqslant |a| - 1 \qquad \text{for } a \notin \{-D, 0, D\}. \tag{3}$$
For the rest of Step 1, we also assume that $|a| \geqslant E = \max\{D + 2, 10\}$. Then by (3) we have $|b| \geqslant D + 1$ and thus $|f(b)| \geqslant D$.

3. Set $c = f(b)$; by (3), we have $|c| \geqslant |b| - 1$. Thus (1) yields
$$a^2 + 4b = c^2 \geqslant \big(|b| - 1\big)^2,$$
which implies
$$a^2 \geqslant \big(|b| - 1\big)^2 - 4|b| = \big(|b| - 3\big)^2 - 8 > \big(|b| - 4\big)^2$$
because $|b| \geqslant |a| - 1 \geqslant 9$. Thus (3) can be refined to
$$|a| + 3 \geqslant |f(a)| \geqslant |a| - 1 \qquad \text{for } |a| \geqslant E.$$
Now, from $c^2 = a^2 + 4b$ with $|b| \in \big[|a| - 1, |a| + 3\big]$ we get $c^2 = (a \pm 2)^2 + d$, where $d \in \{-16, -12, -8, -4, 0, 4, 8\}$. Since $|a \pm 2| \geqslant 8$, this can happen only if $c^2 = (a \pm 2)^2$, which in turn yields $b = \pm a + 1$. To summarise,
$$f(a) = 1 \pm a \qquad \text{for } |a| \geqslant E. \tag{4}$$

We have shown that, with at most finitely many exceptions, $f(a) = 1 \pm a$. Thus it will be convenient for our second step to introduce the sets

$$Z_+ = \big\{a \in \mathbb{Z} \colon f(a) = a + 1\big\}, \quad Z_- = \big\{a \in \mathbb{Z} \colon f(a) = 1 - a\big\}, \quad \text{and} \quad Z_0 = \mathbb{Z} \backslash \big(Z_+ \cup Z_-\big).$$

Step 2. Now we investigate the structure of the sets Z_+, Z_-, and Z_0.

4. Note that $f(E+1) = 1 \pm (E+1)$. If $f(E+1) = E+2$, then $E+1 \in Z_+$. Otherwise we have $f(1+E) = -E$; then the original equation (1) with $n = E+1$ gives us $(E-1)^2 = f(-E)^2$, so $f(-E) = \pm(E-1)$. By (4) this may happen only if $f(-E) = 1 - E$, so in this case $-E \in Z_+$. In any case we find that $Z_+ \neq \varnothing$.

5. Now take any $a \in Z_+$. We claim that every integer $x \geqslant a$ also lies in Z_+. We proceed by induction on x, the base case $x = a$ being covered by our assumption. For the induction step, assume that $f(x-1) = x$ and plug $n = x - 1$ into (1). We get $f(x)^2 = (x+1)^2$, so either $f(x) = x + 1$ or $f(x) = -(x+1)$.

   Assume that $f(x) = -(x+1)$ and $x \neq -1$, since otherwise we already have $f(x) = x+1$. Plugging $n = x$ into (1), we obtain $f(-x-1)^2 = (x-2)^2 - 8$, which may happen only if $x-2 = \pm 3$ and $f(-x-1) = \pm 1$. Plugging $n = -x-1$ into (1), we get $f(\pm 1)^2 = (x+1)^2 \pm 4$, which in turn may happen only if $x + 1 \in \{-2, 0, 2\}$.

   Thus $x \in \{-1, 5\}$ and at the same time $x \in \{-3, -1, 1\}$, which gives us $x = -1$. Since this has already been excluded, we must have $f(x) = x + 1$, which completes our induction.

6. Now we know that either $Z_+ = \mathbb{Z}$ (if Z_+ is not bounded below), or $Z_+ = \{a \in \mathbb{Z} \colon a \geqslant a_0\}$, where a_0 is the smallest element of Z_+. In the former case, $f(n) = n + 1$ for all $n \in \mathbb{Z}$, which is our first solution. So we assume in the following that Z_+ is bounded below and has a smallest element a_0.

   If $Z_0 = \varnothing$, then we have $f(x) = x + 1$ for $x \geqslant a_0$ and $f(x) = 1 - x$ for $x < a_0$. In particular, $f(0) = 1$ in any case, so $0 \in Z_+$ and thus $a_0 \leqslant 0$. Thus we end up with the second solution listed in the answer. It remains to consider the case where $Z_0 \neq \varnothing$.

7. Assume that there exists some $a \in Z_0$ with $b = f(a) \notin Z_0$, so that $f(b) = 1 \pm b$. Then we have $a^2 + 4b = (1 \pm b)^2$, so either $a^2 = (b-1)^2$ or $a^2 = (b-3)^2 - 8$. In the former case we have $b = 1 \pm a$, which is impossible by our choice of a. So we get $a^2 = (b-3)^2 - 8$, which implies $f(b) = 1 - b$ and $|a| = 1$, $|b - 3| = 3$.

   If $b = 0$, then we have $f(b) = 1$, so $b \in Z_+$ and therefore $a_0 \leqslant 0$; hence $a = -1$. But then $f(a) = 0 = a + 1$, so $a \in Z_+$, which is impossible.

   If $b = 6$, then we have $f(6) = -5$, so $f(-5)^2 = 16$ and $f(-5) \in \{-4, 4\}$. Then $f(f(-5))^2 = 25 + 4f(-5) \in \{9, 41\}$, so $f(-5) = -4$ and $-5 \in Z_+$. This implies $a_0 \leqslant -5$, which contradicts our assumption that $\pm 1 = a \notin Z_+$.

8. Thus we have shown that $f(Z_0) \subseteq Z_0$, and Z_0 is finite. Take any element $c \in Z_0$, and consider the sequence defined by $c_i = f^i(c)$. All elements of the sequence (c_i) lie in Z_0, hence it is bounded. Choose an index k for which $|c_k|$ is maximal, so that in particular $|c_{k+1}| \leqslant |c_k|$ and $|c_{k+2}| \leqslant |c_k|$. Our functional equation (1) yields

   $$(|c_k| - 2)^2 - 4 = |c_k|^2 - 4|c_k| \leqslant c_k^2 + 4c_{k+1} = c_{k+2}^2.$$

   Since c_k and c_{k+2} have the same parity and $|c_{k+2}| \leqslant |c_k|$, this leaves us with three possibilities: $|c_{k+2}| = |c_k|$, $|c_{k+2}| = |c_k| - 2$, and $|c_k| - 2 = \pm 2$, $c_{k+2} = 0$.

   If $|c_{k+2}| = |c_k| - 2$, then $f(c_k) = c_{k+1} = 1 - |c_k|$, which means that $c_k \in Z_-$ or $c_k \in Z_+$, and we reach a contradiction.

   If $|c_{k+2}| = |c_k|$, then $c_{k+1} = 0$, thus $c_{k+3}^2 = 4c_{k+2}$. So either $c_{k+3} \neq 0$ or (by maximality of $|c_{k+2}| = |c_k|$) $c_i = 0$ for all i. In the former case, we can repeat the entire argument with c_{k+2} in the place of c_k. Now $|c_{k+4}| = |c_{k+2}|$ is not possible any more since $c_{k+3} \neq 0$, leaving us with the only possibility $|c_k| - 2 = |c_{k+2}| - 2 = \pm 2$.

Thus we know now that either all c_i are equal to 0, or $|c_k| = 4$. If $c_k = \pm 4$, then either $c_{k+1} = 0$ and $|c_{k+2}| = |c_k| = 4$, or $c_{k+2} = 0$ and $c_{k+1} = -4$. From this point onwards, all elements of the sequence are either 0 or ± 4.

Let c_r be the last element of the sequence that is not equal to 0 or ± 4 (if such an element exists). Then $c_{r+1}, c_{r+2} \in \{-4, 0, 4\}$, so

$$c_r^2 = c_{r+2}^2 - 4c_{r+1} \in \{-16, 0, 16, 32\},$$

which gives us a contradiction. Thus all elements of the sequence are equal to 0 or ± 4, and since the choice of $c_0 = c$ was arbitrary, $Z_0 \subseteq \{-4, 0, 4\}$.

9. Finally, we show that $4 \notin Z_0$ and $-4 \notin Z_0$. Suppose that $4 \in Z_0$. Then in particular a_0 (the smallest element of Z_+) cannot be less than 4, since this would imply $4 \in Z_+$. So $-3 \in Z_-$, which means that $f(-3) = 4$. Then $25 = (-3)^2 + 4f(-3) = f(f(-3))^2 = f(4)^2$, so $f(4) = \pm 5 \notin Z_0$, and we reach a contradiction.

   Suppose that $-4 \in Z_0$. The only possible values for $f(-4)$ that are left are 0 and -4. Note that $4f(0) = f(f(0))^2$, so $f(0) \geqslant 0$. If $f(-4) = 0$, then we get $16 = (-4)^2 + 0 = f(0)^2$, thus $f(0) = 4$. But then $f(f(-4)) \notin Z_0$, which is impossible. Thus $f(-4) = -4$, which gives us $0 = (-4)^2 + 4f(-4) = f(f(-4))^2 = 16$, and this is clearly absurd.

   Now we are left with $Z_0 = \{0\}$ and $f(0) = 0$ as the only possibility. If $1 \in Z_-$, then $f(1) = 0$, so $1 = 1^2 + 4f(1) = f(f(1))^2 = f(0)^2 = 0$, which is another contradiction. Thus $1 \in Z_+$, meaning that $a_0 \leqslant 1$. On the other hand, $a_0 \leqslant 0$ would imply $0 \in Z_+$, so we can only have $a_0 = 1$. Thus Z_+ comprises all positive integers, and Z_- comprises all negative integers. This gives us the third solution.

Comment. All solutions known to the Problem Selection Committee are quite lengthy and technical, as the two solutions presented here show. It is possible to make the problem easier by imposing additional assumptions, such as $f(0) \neq 0$ or $f(n) \geqslant 1$ for all $n \geqslant 0$, to remove some of the technicalities.

Combinatorics

C1. Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n+1$ smaller rectangles.

(Serbia)

Solution 1. Let k be the number of rectangles in the dissection. The set of all points that are corners of one of the rectangles can be divided into three disjoint subsets:

- A, which consists of the four corners of the original rectangle R, each of which is the corner of exactly one of the smaller rectangles,

- B, which contains points where exactly two of the rectangles have a common corner (T-junctions, see the figure below),

- C, which contains points where four of the rectangles have a common corner (crossings, see the figure below).

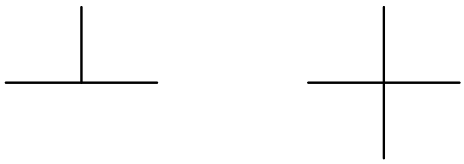


Figure 1: A T-junction and a crossing

We denote the number of points in B by b and the number of points in C by c. Since each of the k rectangles has exactly four corners, we get

$$4k = 4 + 2b + 4c.$$

It follows that $2b \leqslant 4k - 4$, so $b \leqslant 2k - 2$.

Each of the n given points has to lie on a side of one of the smaller rectangles (but not of the original rectangle R). If we extend this side as far as possible along borders between rectangles, we obtain a line segment whose ends are T-junctions. Note that every point in B can only be an endpoint of at most one such segment containing one of the given points, since it is stated that no two of them lie on a common line parallel to the sides of R. This means that

$$b \geqslant 2n.$$

Combining our two inequalities for b, we get

$$2k - 2 \geqslant b \geqslant 2n,$$

thus $k \geqslant n + 1$, which is what we wanted to prove.

Solution 2. Let k denote the number of rectangles. In the following, we refer to the directions of the sides of R as 'horizontal' and 'vertical' respectively. Our goal is to prove the inequality $k \geqslant n+1$ for fixed n. Equivalently, we can prove the inequality $n \leqslant k-1$ for each k, which will be done by induction on k. For $k=1$, the statement is trivial.

Now assume that $k>1$. If none of the line segments that form the borders between the rectangles is horizontal, then we have $k-1$ vertical segments dividing R into k rectangles. On each of them, there can only be one of the n points, so $n \leqslant k-1$, which is exactly what we want to prove.

Otherwise, consider the lowest horizontal line h that contains one or more of these line segments. Let R' be the rectangle that results when everything that lies below h is removed from R (see the example in the figure below).

The rectangles that lie entirely below h form blocks of rectangles separated by vertical line segments. Suppose there are r blocks and k_i rectangles in the i^{th} block. The left and right border of each block has to extend further upwards beyond h. Thus we can move any points that lie on these borders upwards, so that they now lie inside R'. This can be done without violating the conditions, one only needs to make sure that they do not get to lie on a common horizontal line with one of the other given points.

All other borders between rectangles in the i^{th} block have to lie entirely below h. There are k_i-1 such line segments, each of which can contain at most one of the given points. Finally, there can be one point that lies on h. All other points have to lie in R' (after moving some of them as explained in the previous paragraph).

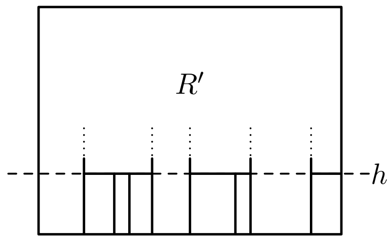


Figure 2: Illustration of the inductive argument

We see that R' is divided into $k-\sum_{i=1}^{r} k_i$ rectangles. Applying the induction hypothesis to R', we find that there are at most

$$\left(k-\sum_{i=1}^{r} k_i\right)-1+\sum_{i=1}^{r}(k_i-1)+1=k-r$$

points. Since $r \geqslant 1$, this means that $n \leqslant k-1$, which completes our induction.

C2. We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a+b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m.

(Iran)

Solution. Let P_k be the product of the numbers on the sheets after k steps.

Suppose that in the $(k+1)^{\text{th}}$ step the numbers a and b are replaced by $a+b$. In the product, the number ab is replaced by $(a+b)^2$, and the other factors do not change. Since $(a+b)^2 \geqslant 4ab$, we see that $P_{k+1} \geqslant 4P_k$. Starting with $P_0 = 1$, a straightforward induction yields

$$P_k \geqslant 4^k$$

for all integers $k \geqslant 0$; in particular

$$P_{m \cdot 2^{m-1}} \geqslant 4^{m \cdot 2^{m-1}} = (2^m)^{2^m},$$

so by the AM–GM inequality, the sum of the numbers written on the sheets after $m2^{m-1}$ steps is at least

$$2^m \cdot \sqrt[2^m]{P_{m \cdot 2^{m-1}}} \geqslant 2^m \cdot 2^m = 4^m .$$

Comment 1. It is possible to achieve the sum 4^m in $m2^{m-1}$ steps. For example, starting from 2^m equal numbers on the sheets, in 2^{m-1} consecutive steps we can double all numbers. After m such doubling rounds we have the number 2^m on every sheet.

Comment 2. There are several versions of the solution above. E.g., one may try to assign to each positive integer n a *weight* w_n in such a way that the sum of the weights of the numbers written on the sheets increases, say, by at least 2 in each step. For this purpose, one needs the inequality

$$2w_{a+b} \geqslant w_a + w_b + 2 \tag{1}$$

to be satisfied for all positive integers a and b.

Starting from $w_1 = 1$ and trying to choose the weights as small as possible, one may find that these weights can be defined as follows: For every positive integer n, one chooses k to be the maximal integer such that $n \geqslant 2^k$, and puts

$$w_n = k + \frac{n}{2^k} = \min_{d \in \mathbb{Z}_{\geqslant 0}} \left(d + \frac{n}{2^d} \right). \tag{2}$$

Now, in order to prove that these weights satisfy (1), one may take arbitrary positive integers a and b, and choose an integer $d \geqslant 0$ such that $w_{a+b} = d + \frac{a+b}{2^d}$. Then one has

$$2w_{a+b} = 2d + 2 \cdot \frac{a+b}{2^d} = \left((d-1) + \frac{a}{2^{d-1}} \right) + \left((d-1) + \frac{b}{2^{d-1}} \right) + 2 \geqslant w_a + w_b + 2.$$

Since the initial sum of the weights was 2^m, after $m2^{m-1}$ steps the sum is at least $(m+1)2^m$. To finish the solution, one may notice that by (2) for every positive integer a one has

$$w_a \leqslant m + \frac{a}{2^m}, \qquad \text{i.e.,} \quad a \geqslant 2^m(-m + w_a). \tag{3}$$

So the sum of the numbers $a_1, a_2, \ldots, a_{2^m}$ on the sheets can be estimated as

$$\sum_{i=1}^{2^m} a_i \geqslant \sum_{i=1}^{2^m} 2^m(-m + w_{a_i}) = -m2^m \cdot 2^m + 2^m \sum_{i=1}^{2^m} w_{a_i} \geqslant -m4^m + (m+1)4^m = 4^m,$$

as required.

For establishing the inequalities (1) and (3), one may also use the convexity argument, instead of the second definition of w_n in (2).

One may check that $\log_2 n \leqslant w_n \leqslant \log_2 n + 1$; thus, in some rough sense, this approach is obtained by "taking the logarithm" of the solution above.

Comment 3. An intuitive strategy to minimise the sum of numbers is that in every step we choose the two smallest numbers. We may call this the *greedy strategy.* In the following paragraphs we prove that the greedy strategy indeed provides the least possible sum of numbers.

Claim. Starting from any sequence $x_1, \dots, x_N$ of positive real numbers on N sheets, for any number k of steps, the greedy strategy achieves the lowest possible sum of numbers.

Proof. We apply induction on k; for $k = 1$ the statement is obvious. Let $k \geqslant 2$, and assume that the claim is true for smaller values.

Every sequence of k steps can be encoded as $S = \big((i_1, j_1), \dots, (i_k, j_k)\big)$, where, for $r = 1, 2, \dots, k$, the numbers i_r and j_r are the indices of the two sheets that are chosen in the r^{th} step. The resulting final sum will be some linear combination of $x_1, \dots, x_N$, say, $c_1x_1 + \dots + c_Nx_N$ with positive integers $c_1, \dots, c_N$ that depend on S only. Call the numbers $(c_1, \dots, c_N)$ the *characteristic vector* of S.

Choose a sequence $S_0 = \big((i_1, j_1), \dots, (i_k, j_k)\big)$ of steps that produces the minimal sum, starting from $x_1, \dots, x_N$, and let $(c_1, \dots, c_N)$ be the characteristic vector of S. We may assume that the sheets are indexed in such an order that $c_1 \geqslant c_2 \geqslant \dots \geqslant c_N$. If the sheets (and the numbers) are permuted by a permutation π of the indices $(1, 2, \dots, N)$ and then the same steps are performed, we can obtain the sum $\sum\limits_{t=1}^{N} c_t x_{\pi(t)}$. By the rearrangement inequality, the smallest possible sum can be achieved when the numbers $(x_1, \dots, x_N)$ are in non-decreasing order. So we can assume that also $x_1 \leqslant x_2 \leqslant \dots \leqslant x_N$.

Let ℓ be the largest index with $c_1 = \dots = c_\ell$, and let the r^{th} step be the first step for which $c_{i_r} = c_1$ or $c_{j_r} = c_1$. The role of i_r and j_r is symmetrical, so we can assume $c_{i_r} = c_1$ and thus $i_r \leqslant \ell$. We show that $c_{j_r} = c_1$ and $j_r \leqslant \ell$ hold, too.

Before the r^{th} step, on the ${i_r}^{\text{th}}$ sheet we had the number x_{i_r}. On the ${j_r}^{\text{th}}$ sheet there was a linear combination that contains the number x_{j_r} with a positive integer coefficient, and possibly some other terms. In the r^{th} step, the number x_{i_r} joins that linear combination. From this point, each sheet contains a linear combination of $x_1, \dots, x_N$, with the coefficient of x_{j_r} being not smaller than the coefficient of x_{i_r}. This is preserved to the end of the procedure, so we have $c_{j_r} \geqslant c_{i_r}$. But $c_{i_r} = c_1$ is maximal among the coefficients, so we have $c_{j_r} = c_{i_r} = c_1$ and thus $j_r \leqslant \ell$.

Either from $c_{j_r} = c_{i_r} = c_1$ or from the arguments in the previous paragraph we can see that none of the ${i_r}^{\text{th}}$ and the ${j_r}^{\text{th}}$ sheets were used before step r. Therefore, the final linear combination of the numbers does not change if the step (i_r, j_r) is performed first: the sequence of steps

$$S_1 = \big((i_r, j_r), (i_1, j_1), \dots, (i_{r-1}, j_{r-1}), (i_{r+1}, j_{r+1}), \dots, (i_N, j_N)\big)$$

also produces the same minimal sum at the end. Therefore, we can replace S_0 by S_1 and we may assume that $r = 1$ and $c_{i_1} = c_{j_1} = c_1$.

As $i_1 \neq j_1$, we can see that $\ell \geqslant 2$ and $c_1 = c_2 = c_{i_1} = c_{j_1}$. Let π be such a permutation of the indices $(1, 2, \dots, N)$ that exchanges $1, 2$ with i_r, j_r and does not change the remaining indices. Let

$$S_2 = \big((\pi(i_1), \pi(j_1)), \dots, (\pi(i_N), \pi(j_N))\big).$$

Since $c_{\pi(i)} = c_i$ for all indices i, this sequence of steps produces the same, minimal sum. Moreover, in the first step we chose $x_{\pi(i_1)} = x_1$ and $x_{\pi(j_1)} = x_2$, the two smallest numbers.

Hence, it is possible to achieve the optimal sum if we follow the greedy strategy in the first step. By the induction hypothesis, following the greedy strategy in the remaining steps we achieve the optimal sum.

C3. Let $n \geqslant 2$ be an integer. Consider an $n \times n$ chessboard divided into n^2 unit squares. We call a configuration of n rooks on this board *happy* if every row and every column contains exactly one rook. Find the greatest positive integer k such that for every happy configuration of rooks, we can find a $k \times k$ square without a rook on any of its k^2 unit squares.

(Croatia)

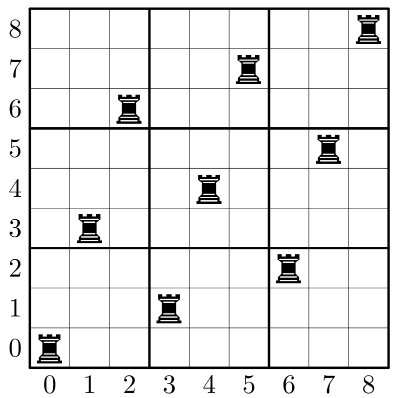
Answer. $\left\lfloor \sqrt{n-1} \right\rfloor$.

Solution. Let ℓ be a positive integer. We will show that **(i)** if $n > \ell^2$ then each happy configuration contains an empty $\ell \times \ell$ square, but **(ii)** if $n \leqslant \ell^2$ then there exists a happy configuration not containing such a square. These two statements together yield the answer.

(i). Assume that $n > \ell^2$. Consider any happy configuration. There exists a row R containing a rook in its leftmost square. Take ℓ consecutive rows with R being one of them. Their union U contains exactly ℓ rooks. Now remove the $n - \ell^2 \geqslant 1$ leftmost columns from U (thus at least one rook is also removed). The remaining part is an $\ell^2 \times \ell$ rectangle, so it can be split into ℓ squares of size $\ell \times \ell$, and this part contains at most $\ell - 1$ rooks. Thus one of these squares is empty.

(ii). Now we assume that $n \leqslant \ell^2$. Firstly, we will construct a happy configuration with no empty $\ell \times \ell$ square for the case $n = \ell^2$. After that we will modify it to work for smaller values of n.

Let us enumerate the rows from bottom to top as well as the columns from left to right by the numbers $0, 1, \ldots, \ell^2 - 1$. Every square will be denoted, as usual, by the pair (r, c) of its row and column numbers. Now we put the rooks on all squares of the form $(i\ell + j, j\ell + i)$ with $i, j = 0, 1, \ldots, \ell - 1$ (the picture below represents this arrangement for $\ell = 3$). Since each number from 0 to $\ell^2 - 1$ has a unique representation of the form $i\ell + j$ $(0 \leqslant i, j \leqslant \ell - 1)$, each row and each column contains exactly one rook.



Next, we show that each $\ell \times \ell$ square A on the board contains a rook. Consider such a square A, and consider ℓ consecutive rows the union of which contains A. Let the lowest of these rows have number $p\ell + q$ with $0 \leqslant p, q \leqslant \ell - 1$ (notice that $p\ell + q \leqslant \ell^2 - \ell$). Then the rooks in this union are placed in the columns with numbers $q\ell + p$, $(q+1)\ell + p$, $\ldots$, $(\ell - 1)\ell + p$, $p + 1$, $\ell + (p+1)$, $\ldots$, $(q-1)\ell + p + 1$, or, putting these numbers in increasing order,

$$p + 1,\ \ell + (p+1),\ \ldots,\ (q-1)\ell + (p+1),\ q\ell + p,\ (q+1)\ell + p,\ \ldots,\ (\ell - 1)\ell + p.$$

One readily checks that the first number in this list is at most $\ell - 1$ (if $p = \ell - 1$, then $q = 0$, and the first listed number is $q\ell + p = \ell - 1$), the last one is at least $(\ell - 1)\ell$, and the difference between any two consecutive numbers is at most ℓ. Thus, one of the ℓ consecutive columns intersecting A contains a number listed above, and the rook in this column is inside A, as required. The construction for $n = \ell^2$ is established.

It remains to construct a happy configuration of rooks not containing an empty $\ell \times \ell$ square for $n < \ell^2$. In order to achieve this, take the construction for an $\ell^2 \times \ell^2$ square described above and remove the $\ell^2 - n$ bottom rows together with the $\ell^2 - n$ rightmost columns. We will have a rook arrangement with no empty $\ell \times \ell$ square, but several rows and columns may happen to be empty. Clearly, the number of empty rows is equal to the number of empty columns, so one can find a bijection between them, and put a rook on any crossing of an empty row and an empty column corresponding to each other.

Comment. Part **(i)** allows several different proofs. E.g., in the last paragraph of the solution, it suffices to deal only with the case $n = \ell^2 + 1$. Notice now that among the four corner squares, at least one is empty. So the rooks in its row and in its column are distinct. Now, deleting this row and column we obtain an $\ell^2 \times \ell^2$ square with $\ell^2 - 1$ rooks in it. This square can be partitioned into ℓ^2 squares of size $\ell \times \ell$, so one of them is empty.

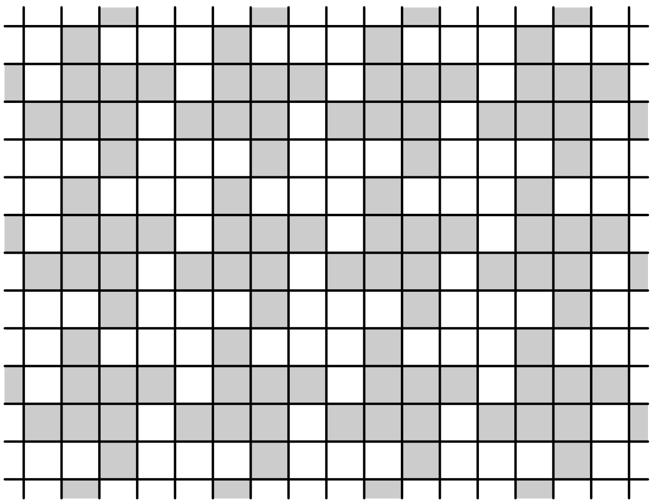
C4. Construct a tetromino by attaching two 2×1 dominoes along their longer sides such that the midpoint of the longer side of one domino is a corner of the other domino. This construction yields two kinds of tetrominoes with opposite orientations. Let us call them S- and Z-tetrominoes, respectively.

S-tetrominoes Z-tetrominoes

Assume that a lattice polygon P can be tiled with S-tetrominoes. Prove than no matter how we tile P using only S- and Z-tetrominoes, we always use an even number of Z-tetrominoes.

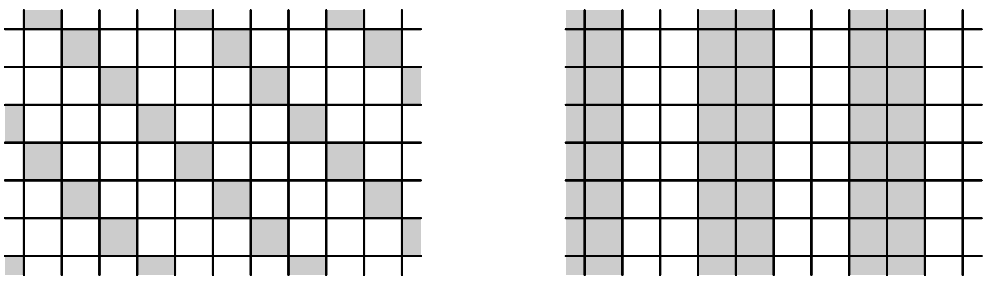
(Hungary)

Solution 1. We may assume that polygon P is the union of some squares of an infinite chessboard. Colour the squares of the chessboard with two colours as the figure below illustrates.



Observe that no matter how we tile P, any S-tetromino covers an even number of black squares, whereas any Z-tetromino covers an odd number of them. As P can be tiled exclusively by S-tetrominoes, it contains an even number of black squares. But if some S-tetrominoes and some Z-tetrominoes cover an even number of black squares, then the number of Z-tetrominoes must be even.

Comment. An alternative approach makes use of the following two colourings, which are perhaps somewhat more natural:



Let s_1 and s_2 be the number of S-tetrominoes of the first and second type (as shown in the figure above) respectively that are used in a tiling of P. Likewise, let z_1 and z_2 be the number of Z-tetrominoes of the first and second type respectively. The first colouring shows that $s_1 + z_2$ is invariant modulo 2, the second colouring shows that $s_1 + z_1$ is invariant modulo 2. Adding these two conditions, we find that $z_1 + z_2$ is invariant modulo 2, which is what we have to prove. Indeed, the sum of the two colourings (regarding white as 0 and black as 1 and adding modulo 2) is the colouring shown in the solution.

Solution 2. Let us assign coordinates to the squares of the infinite chessboard in such a way that the squares of P have nonnegative coordinates only, and that the first coordinate increases as one moves to the right, while the second coordinate increases as one moves upwards. Write the integer $3^i \cdot (-3)^j$ into the square with coordinates (i, j), as in the following figure:

$\vdots$					
81	$\vdots$				
-27	-81	$\vdots$			
9	27	81	$\cdots$		
-3	-9	-27	-81	$\cdots$	
1	3	9	27	81	$\cdots$

The sum of the numbers written into four squares that can be covered by an S-tetromino is either of the form

$$3^i \cdot (-3)^j \cdot \big(1 + 3 + 3 \cdot (-3) + 3^2 \cdot (-3)\big) = -32 \cdot 3^i \cdot (-3)^j$$

(for the first type of S-tetrominoes), or of the form

$$3^i \cdot (-3)^j \cdot \big(3 + 3 \cdot (-3) + (-3) + (-3)^2\big) = 0,$$

and thus divisible by 32. For this reason, the sum of the numbers written into the squares of P, and thus also the sum of the numbers covered by Z-tetrominoes in the second covering, is likewise divisible by 32. Now the sum of the entries of a Z-tetromino is either of the form

$$3^i \cdot (-3)^j \cdot \big(3 + 3^2 + (-3) + 3 \cdot (-3)\big) = 0$$

(for the first type of Z-tetrominoes), or of the form

$$3^i \cdot (-3)^j \cdot \big(1 + (-3) + 3 \cdot (-3) + 3 \cdot (-3)^2\big) = 16 \cdot 3^i \cdot (-3)^j,$$

i.e., 16 times an odd number. Thus in order to obtain a total that is divisible by 32, an even number of the latter kind of Z-tetrominoes needs to be used. Rotating everything by 90°, we find that the number of Z-tetrominoes of the first kind is even as well. So we have even proven slightly more than necessary.

Comment 1. In the second solution, 3 and -3 can be replaced by other combinations as well. For example, for any positive integer $a \equiv 3 \pmod 4$, we can write $a^i \cdot (-a)^j$ into the square with coordinates (i, j) and apply the same argument.

Comment 2. As the second solution shows, we even have the stronger result that the parity of the number of each of the four types of tetrominoes in a tiling of P by S- and Z-tetrominoes is an invariant of P. This also remains true if there is no tiling of P that uses only S-tetrominoes.

C5. Consider $n \geqslant 3$ lines in the plane such that no two lines are parallel and no three have a common point. These lines divide the plane into polygonal regions; let $\mathcal{F}$ be the set of regions having finite area. Prove that it is possible to colour $\lceil \sqrt{n/2} \rceil$ of the lines blue in such a way that no region in $\mathcal{F}$ has a completely blue boundary. (For a real number x, $\lceil x \rceil$ denotes the least integer which is not smaller than x.)

(Austria)

Solution. Let L be the given set of lines. Choose a maximal (by inclusion) subset $B \subseteq L$ such that when we colour the lines of B blue, no region in $\mathcal{F}$ has a completely blue boundary. Let $|B| = k$. We claim that $k \geqslant \lceil \sqrt{n/2} \rceil$.

Let us colour all the lines of $L \backslash B$ red. Call a point *blue* if it is the intersection of two blue lines. Then there are $\binom{k}{2}$ blue points.

Now consider any red line ℓ. By the maximality of B, there exists at least one region $A \in \mathcal{F}$ whose only red side lies on ℓ. Since A has at least three sides, it must have at least one blue vertex. Let us take one such vertex and associate it to ℓ.

Since each blue point belongs to four regions (some of which may be unbounded), it is associated to at most four red lines. Thus the total number of red lines is at most $4\binom{k}{2}$. On the other hand, this number is $n-k$, so

$$n - k \leqslant 2k(k-1), \quad \text{thus} \quad n \leqslant 2k^2 - k \leqslant 2k^2,$$

and finally $k \geqslant \lceil \sqrt{n/2} \rceil$, which gives the desired result.

Comment 1. The constant factor in the estimate can be improved in different ways; we sketch two of them below. On the other hand, the Problem Selection Committee is not aware of any results showing that it is sometimes impossible to colour k lines satisfying the desired condition for $k \gg \sqrt{n}$. In this situation we find it more suitable to keep the original formulation of the problem.

1. Firstly, we show that in the proof above one has in fact $k = |B| \geqslant \lceil \sqrt{2n/3} \rceil$.

Let us make weighted associations as follows. Let a region A whose only red side lies on ℓ have k vertices, so that $k-2$ of them are blue. We associate each of these blue vertices to ℓ, and put the weight $\frac{1}{k-2}$ on each such association. So the sum of the weights of all the associations is exactly $n-k$.

Now, one may check that among the four regions adjacent to a blue vertex v, at most two are triangles. This means that the sum of the weights of all associations involving v is at most $1 + 1 + \frac{1}{2} + \frac{1}{2} = 3$. This leads to the estimate

$$n - k \leqslant 3\binom{k}{2},$$

or

$$2n \leqslant 3k^2 - k < 3k^2,$$

which yields $k \geqslant \lceil \sqrt{2n/3} \rceil$.

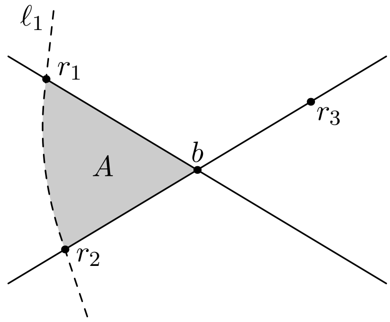
2. Next, we even show that $k = |B| \geqslant \lceil \sqrt{n} \rceil$. For this, we specify the process of associating points to red lines in one more different way.

Call a point *red* if it lies on a red line as well as on a blue line. Consider any red line ℓ, and take an arbitrary region $A \in \mathcal{F}$ whose only red side lies on ℓ. Let r', r, $b_1, \ldots, b_k$ be its vertices in clockwise order with $r', r \in \ell$; then the points r', r are red, while all the points $b_1, \ldots, b_k$ are blue. Let us associate to ℓ the red point r and the blue point b_1. One may notice that to each pair of a red point r and a blue point b, at most one red line can be associated, since there is at most one region A having r and b as two clockwise consecutive vertices.

We claim now that at most two red lines are associated to each blue point b; this leads to the desired bound

$$n - k \leqslant 2\binom{k}{2} \quad \Longleftrightarrow \quad n \leqslant k^2.$$

Assume, to the contrary, that three red lines ℓ_1, ℓ_2, and ℓ_3 are associated to the same blue point b. Let r_1, r_2, and r_3 respectively be the red points associated to these lines; all these points are distinct. The point b defines four blue rays, and each point r_i is the red point closest to b on one of these rays. So we may assume that the points r_2 and r_3 lie on one blue line passing through b, while r_1 lies on the other one.



Now consider the region A used to associate r_1 and b with ℓ_1. Three of its clockwise consecutive vertices are r_1, b, and either r_2 or r_3 (say, r_2). Since A has only one red side, it can only be the triangle r_1br_2; but then both ℓ_1 and ℓ_2 pass through r_2, as well as some blue line. This is impossible by the problem assumptions.

Comment 2. The condition that the lines be non-parallel is essentially not used in the solution, nor in the previous comment; thus it may be omitted.

C6. We are given an infinite deck of cards, each with a real number on it. For every real number x, there is exactly one card in the deck that has x written on it. Now two players draw disjoint sets A and B of 100 cards each from this deck. We would like to define a rule that declares one of them a winner. This rule should satisfy the following conditions:

1. The winner only depends on the relative order of the 200 cards: if the cards are laid down in increasing order face down and we are told which card belongs to which player, but not what numbers are written on them, we can still decide the winner.

2. If we write the elements of both sets in increasing order as $A = \{a_1, a_2, \ldots, a_{100}\}$ and $B = \{b_1, b_2, \ldots, b_{100}\}$, and $a_i > b_i$ for all i, then A beats B.

3. If three players draw three disjoint sets A, B, C from the deck, A beats B and B beats C, then A also beats C.

How many ways are there to define such a rule? Here, we consider two rules as different if there exist two sets A and B such that A beats B according to one rule, but B beats A according to the other.

(Russia)

Answer. 100.

Solution 1. We prove a more general statement for sets of cardinality n (the problem being the special case $n = 100$, then the answer is n). In the following, we write $A > B$ or $B < A$ for "A beats B".

Part I. Let us first define n different rules that satisfy the conditions. To this end, fix an index $k \in \{1, 2, \ldots, n\}$. We write both A and B in increasing order as $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ and say that A beats B if and only if $a_k > b_k$. This rule clearly satisfies all three conditions, and the rules corresponding to different k are all different. Thus there are at least n different rules.

Part II. Now we have to prove that there is no other way to define such a rule. Suppose that our rule satisfies the conditions, and let $k \in \{1, 2, \ldots, n\}$ be minimal with the property that

$$A_k = \{1, 2, \ldots, k, n+k+1, n+k+2, \ldots, 2n\} \prec B_k = \{k+1, k+2, \ldots, n+k\}.$$

Clearly, such a k exists, since this holds for $k = n$ by assumption. Now consider two disjoint sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, both in increasing order (i.e., $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$). We claim that $X \prec Y$ if (and only if – this follows automatically) $x_k < y_k$.

To prove this statement, pick arbitrary real numbers $u_i, v_i, w_i \notin X \cup Y$ such that

$$u_1 < u_2 < \cdots < u_{k-1} < \min(x_1, y_1), \quad \max(x_n, y_n) < v_{k+1} < v_{k+2} < \cdots < v_n,$$

and

$$x_k < v_1 < v_2 < \cdots < v_k < w_1 < w_2 < \cdots < w_n < u_k < u_{k+1} < \cdots < u_n < y_k,$$

and set

$$U = \{u_1, u_2, \ldots, u_n\}, \; V = \{v_1, v_2, \ldots, v_n\}, \; W = \{w_1, w_2, \ldots, w_n\}.$$

Then

- $u_i < y_i$ and $x_i < v_i$ for all i, so $U \prec Y$ and $X \prec V$ by the second condition.

- The elements of $U \cup W$ are ordered in the same way as those of $A_{k-1} \cup B_{k-1}$, and since $A_{k-1} \succ B_{k-1}$ by our choice of k, we also have $U \succ W$ (if $k = 1$, this is trivial).

- The elements of $V \cup W$ are ordered in the same way as those of $A_k \cup B_k$, and since $A_k \prec B_k$ by our choice of k, we also have $V \prec W$.

It follows that
$$X \prec V \prec W \prec U \prec Y,$$
so $X \prec Y$ by the third condition, which is what we wanted to prove.

Solution 2. Another possible approach to Part II of this problem is induction on n. For $n = 1$, there is trivially only one rule in view of the second condition.

In the following, we assume that our claim (namely, that there are no possible rules other than those given in Part I) holds for $n-1$ in place of n. We start with the following observation:

Claim. At least one of the two relations
$$\big(\{2\} \cup \{2i-1 \,|\, 2 \leqslant i \leqslant n\}\big) \prec \big(\{1\} \cup \{2i \,|\, 2 \leqslant i \leqslant n\}\big)$$
and
$$\big(\{2i-1 \,|\, 1 \leqslant i \leqslant n-1\} \cup \{2n\}\big) \prec \big(\{2i \,|\, 1 \leqslant i \leqslant n-1\} \cup \{2n-1\}\big)$$
holds.

Proof. Suppose that the first relation does not hold. Since our rule may only depend on the relative order, we must also have
$$\big(\{2\} \cup \{3i-2 \,|\, 2 \leqslant i \leqslant n-1\} \cup \{3n-2\}\big) \succ \big(\{1\} \cup \{3i-1 \,|\, 2 \leqslant i \leqslant n-1\} \cup \{3n\}\big).$$
Likewise, if the second relation does not hold, then we must also have
$$\big(\{1\} \cup \{3i-1 \,|\, 2 \leqslant i \leqslant n-1\} \cup \{3n\}\big) \succ \big(\{3\} \cup \{3i \,|\, 2 \leqslant i \leqslant n-1\} \cup \{3n-1\}\big).$$
Now condition 3 implies that
$$\big(\{2\} \cup \{3i-2 \,|\, 2 \leqslant i \leqslant n-1\} \cup \{3n-2\}\big) \succ \big(\{3\} \cup \{3i \,|\, 2 \leqslant i \leqslant n-1\} \cup \{3n-1\}\big),$$
which contradicts the second condition. □

Now we distinguish two cases, depending on which of the two relations actually holds:

First case: $\big(\{2\} \cup \{2i-1 \,|\, 2 \leqslant i \leqslant n\}\big) \prec \big(\{1\} \cup \{2i \,|\, 2 \leqslant i \leqslant n\}\big)$.

Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be two disjoint sets, both in increasing order. We claim that the winner can be decided only from the values of $a_2, \ldots, a_n$ and $b_2, \ldots, b_n$, while a_1 and b_1 are actually irrelevant. Suppose that this was not the case, and assume without loss of generality that $a_2 < b_2$. Then the relative order of $a_1, a_2, \ldots, a_n, b_2, \ldots, b_n$ is fixed, and the position of b_1 has to decide the winner. Suppose that for some value $b_1 = x$, B wins, while for some other value $b_1 = y$, A wins.

Write $B_x = \{x, b_2, \ldots, b_n\}$ and $B_y = \{y, b_2, \ldots, b_n\}$, and let $\varepsilon > 0$ be smaller than half the distance between any two of the numbers in $B_x \cup B_y \cup A$. For any set M, let $M \pm \varepsilon$ be the set obtained by adding/subtracting ε to all elements of M. By our choice of ε, the relative order of the elements of $(B_y + \varepsilon) \cup A$ is still the same as for $B_y \cup A$, while the relative order of the elements of $(B_x - \varepsilon) \cup A$ is still the same as for $B_x \cup A$. Thus $A \prec B_x - \varepsilon$, but $A \succ B_y + \varepsilon$. Moreover, if $y > x$, then $B_x - \varepsilon \prec B_y + \varepsilon$ by condition 2, while otherwise the relative order of the elements in $(B_x - \varepsilon) \cup (B_y + \varepsilon)$ is the same as for the two sets $\{2\} \cup \{2i-1 \mid 2 \leqslant i \leqslant n\}$ and $\{1\} \cup \{2i \mid 2 \leqslant i \leqslant n\}$, so that $B_x - \varepsilon \prec B_y + \varepsilon$. In either case, we obtain

$$A \prec B_x - \varepsilon \prec B_y + \varepsilon \prec A,$$

which contradicts condition 3.

So we know now that the winner does not depend on a_1, b_1. Therefore, we can define a new rule $\prec^*$ on sets of cardinality $n-1$ by saying that $A \prec^* B$ if and only if $A \cup \{a\} \prec B \cup \{b\}$ for some a, b (or equivalently, all a, b) such that $a < \min A$, $b < \min B$ and $A \cup \{a\}$ and $B \cup \{b\}$ are disjoint. The rule $\prec^*$ satisfies all conditions again, so by the induction hypothesis, there exists an index i such that $A \prec^* B$ if and only if the i^{th} smallest element of A is less than the i^{th} smallest element of B. This implies that $C \prec D$ if and only if the $(i+1)^{\text{th}}$ smallest element of C is less than the $(i+1)^{\text{th}}$ smallest element of D, which completes our induction.

Second case: $\big(\{2i-1 \mid 1 \leqslant i \leqslant n-1\} \cup \{2n\}\big) \prec \big(\{2i \mid 1 \leqslant i \leqslant n-1\} \cup \{2n-1\}\big)$.
Set $-A = \{-a \mid a \in A\}$ for any $A \subseteq \mathbb{R}$. For any two disjoint sets $A, B \subseteq \mathbb{R}$ of cardinality n, we write $A \prec^\circ B$ to mean $(-B) \prec (-A)$. It is easy to see that $\prec^\circ$ defines a rule to determine a winner that satisfies the three conditions of our problem as well as the relation of the first case. So it follows in the same way as in the first case that for some i, $A \prec^\circ B$ if and only if the i^{th} smallest element of A is less than the i^{th} smallest element of B, which is equivalent to the condition that the i^{th} largest element of $-A$ is greater than the i^{th} largest element of $-B$. This proves that the original rule $\prec$ also has the desired form.

Comment. The problem asks for all possible partial orders on the set of n-element subsets of $\mathbb{R}$ such that any two disjoint sets are comparable, the order relation only depends on the relative order of the elements, and $\{a_1, a_2, \ldots, a_n\} \prec \{b_1, b_2, \ldots, b_n\}$ whenever $a_i < b_i$ for all i.

As the proposer points out, one may also ask for all **total** orders on all n-element subsets of $\mathbb{R}$ (dropping the condition of disjointness and requiring that $\{a_1, a_2, \ldots, a_n\} \preceq \{b_1, b_2, \ldots, b_n\}$ whenever $a_i \leqslant b_i$ for all i). It turns out that the number of possibilities in this case is $n!$, and all possible total orders are obtained in the following way. Fix a permutation $\sigma \in S_n$. Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be two subsets of $\mathbb{R}$ with $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_n$. Then we say that $A \succ_\sigma B$ if and only if $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ is lexicographically greater than $(b_{\sigma(1)}, \ldots, b_{\sigma(n)})$.

It seems, however, that this formulation adds rather more technicalities to the problem than additional ideas.

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C7. Let M be a set of $n \geqslant 4$ points in the plane, no three of which are collinear. Initially these points are connected with n segments so that each point in M is the endpoint of exactly two segments. Then, at each step, one may choose two segments AB and CD sharing a common interior point and replace them by the segments AC and BD if none of them is present at this moment. Prove that it is impossible to perform $n^3/4$ or more such moves.

(Russia)

Solution. A line is said to be *red* if it contains two points of M. As no three points of M are collinear, each red line determines a unique pair of points of M. Moreover, there are precisely $\binom{n}{2} < \frac{n^2}{2}$ red lines. By the *value of a segment* we mean the number of red lines intersecting it in its interior, and the *value of a set of segments* is defined to be the sum of the values of its elements. We will prove that (i) the value of the initial set of segments is smaller than $n^3/2$ and that (ii) each step decreases the value of the set of segments present by at least 2. Since such a value can never be negative, these two assertions imply the statement of the problem.

To show (i) we just need to observe that each segment has a value that is smaller than $n^2/2$. Thus the combined value of the n initial segments is indeed below $n \cdot n^2/2 = n^3/2$.

It remains to establish (ii). Suppose that at some moment we have two segments AB and CD sharing an interior point S, and that at the next moment we have the two segments AC and BD instead. Let X_{AB} denote the set of red lines intersecting the segment AB in its interior and let the sets X_{AC}, X_{BD}, and X_{CD} be defined similarly. We are to prove that $|X_{AC}| + |X_{BD}| + 2 \leqslant |X_{AB}| + |X_{CD}|$.

As a first step in this direction, we claim that

$$|X_{AC} \cup X_{BD}| + 2 \leqslant |X_{AB} \cup X_{CD}| . \qquad (1)$$

Indeed, if g is a red line intersecting, e.g. the segment AC in its interior, then it has to intersect the triangle ACS once again, either in the interior of its side AS, or in the interior of its side CS, or at S, meaning that it belongs to X_{AB} or to X_{CD} (see Figure 1). Moreover, the red lines AB and CD contribute to $X_{AB} \cup X_{CD}$ but not to $X_{AC} \cup X_{BD}$. Thereby (1) is proved.

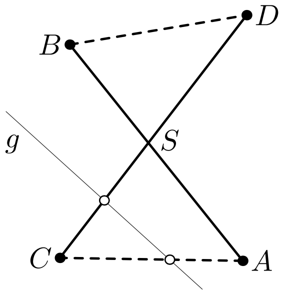


Figure 1

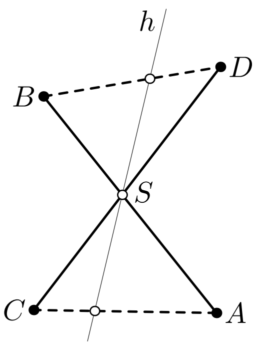


Figure 2

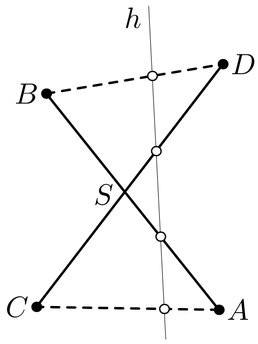


Figure 3

Similarly but more easily one obtains

$$|X_{AC} \cap X_{BD}| \leqslant |X_{AB} \cap X_{CD}| . \qquad (2)$$

Indeed, a red line h appearing in $X_{AC} \cap X_{BD}$ belongs, for similar reasons as above, also to $X_{AB} \cap X_{CD}$. To make the argument precise, one may just distinguish the cases $S \in h$ (see Figure 2) and $S \notin h$ (see Figure 3). Thereby (2) is proved.

Adding (1) and (2) we obtain the desired conclusion, thus completing the solution of this problem.

Comment 1. There is a problem belonging to the folklore, in the solution of which one may use the same kind of operation:

Given n red and n green points in the plane, prove that one may draw n nonintersecting segments each of which connects a red point with a green point.

A standard approach to this problem consists in taking n arbitrary segments connecting the red points with the green points, and to perform the same operation as in the above proposal whenever an intersection occurs. Now each time one performs such a step, the total length of the segments that are present decreases due to the triangle inequality. So, as there are only finitely many possibilities for the set of segments present, the process must end at some stage.

In the above proposal, however, considering the sum of the Euclidean lengths of the segment that are present does not seem to help much, for even though it shows that the process must necessarily terminate after some finite number of steps, it does not seem to easily yield any upper bound on the number of these steps that grows polynomially with n.

One may regard the concept of the value of a segment introduced in the above solution as an appropriately discretised version of Euclidean length suitable for obtaining such a bound.

The Problem Selection Committee still believes the problem to be sufficiently original for the competition.

Comment 2. There are some other essentially equivalent ways of presenting the same solution. E.g., put $M = \{A_1, A_2, \ldots, A_n\}$, denote the set of segments present at any moment by $\{e_1, e_2, \ldots, e_n\}$, and called a triple (i, j, k) of indices with $i \neq j$ *intersecting*, if the line A_iA_j intersects the segment e_k. It may then be shown that the number S of intersecting triples satisfies $0 \leqslant S < n^3$ at the beginning and decreases by at least 4 in each step.

Comment 3. It is not difficult to construct an example where cn^2 moves are possible (for some absolute constant $c > 0$). It would be interesting to say more about the gap between cn^2 and cn^3.

C8. A card deck consists of 1024 cards. On each card, a set of distinct decimal digits is written in such a way that no two of these sets coincide (thus, one of the cards is empty). Two players alternately take cards from the deck, one card per turn. After the deck is empty, each player checks if he can throw out one of his cards so that each of the ten digits occurs on an even number of his remaining cards. If one player can do this but the other one cannot, the one who can is the winner; otherwise a draw is declared.

Determine all possible first moves of the first player after which he has a winning strategy.

(Russia)

Answer. All the moves except for taking the empty card.

Solution. Let us identify each card with the set of digits written on it. For any collection of cards $C_1, C_2, \dots, C_k$ denote by their *sum* the set $C_1 \triangle C_2 \triangle \cdots \triangle C_k$ consisting of all elements belonging to an odd number of the C_i's. Denote the first and the second player by $\mathcal{F}$ and $\mathcal{S}$, respectively.

Since each digit is written on exactly 512 cards, the sum of all the cards is $\varnothing$. Therefore, at the end of the game the sum of all the cards of $\mathcal{F}$ will be the same as that of $\mathcal{S}$; denote this sum by C. Then the player who took C can throw it out and get the desired situation, while the other one cannot. Thus, the player getting card C wins, and no draw is possible.

Now, given a nonempty card B, one can easily see that all the cards can be split into 512 pairs of the form $(X, X \triangle B)$ because $(X \triangle B) \triangle B = X$. The following lemma shows a property of such a partition that is important for the solution.

Lemma. Let $B \neq \varnothing$ be some card. Let us choose 512 cards so that exactly one card is chosen from every pair $(X, X \triangle B)$. Then the sum of all chosen cards is either $\varnothing$ or B.

Proof. Let b be some element of B. Enumerate the pairs; let X_i be the card not containing b in the i^{th} pair, and let Y_i be the other card in this pair. Then the sets X_i are exactly all the sets not containing b, therefore each digit $a \neq b$ is written on exactly 256 of these cards, so $X_1 \triangle X_2 \triangle \cdots \triangle X_{512} = \varnothing$. Now, if we replace some summands in this sum by the other elements from their pairs, we will simply add B several times to this sum, thus the sum will either remain unchanged or change by B, as required. □

Now we consider two cases.

Case 1. Assume that $\mathcal{F}$ takes the card $\varnothing$ on his first move. In this case, we present a winning strategy for $\mathcal{S}$.

Let $\mathcal{S}$ take an arbitrary card A. Assume that $\mathcal{F}$ takes card B after that; then $\mathcal{S}$ takes $A \triangle B$. Split all 1024 cards into 512 pairs of the form $(X, X \triangle B)$; we call two cards in one pair *partners*. Then the four cards taken so far form two pairs $(\varnothing, B)$ and $(A, A \triangle B)$ belonging to $\mathcal{F}$ and $\mathcal{S}$, respectively. On each of the subsequent moves, when $\mathcal{F}$ takes some card, $\mathcal{S}$ should take the partner of this card in response.

Consider the situation at the end of the game. Let us for a moment replace card A belonging to $\mathcal{S}$ by $\varnothing$. Then he would have one card from each pair; by our lemma, the sum of all these cards would be either $\varnothing$ or B. Now, replacing $\varnothing$ back by A we get that the actual sum of the cards of $\mathcal{S}$ is either A or $A \triangle B$, and he has both these cards. Thus $\mathcal{S}$ wins.

Case 2. Now assume that $\mathcal{F}$ takes some card $A \neq \varnothing$ on his first move. Let us present a winning strategy for $\mathcal{F}$ in this case.

Assume that $\mathcal{S}$ takes some card $B \neq \varnothing$ on his first move; then $\mathcal{F}$ takes $A \triangle B$. Again, let us split all the cards into pairs of the form $(X, X \triangle B)$; then the cards which have not been taken yet form several complete pairs and one extra element (card $\varnothing$ has not been taken while its partner B has). Now, on each of the subsequent moves, if $\mathcal{S}$ takes some element from a complete pair, then $\mathcal{F}$ takes its partner. If $\mathcal{S}$ takes the extra element, then $\mathcal{F}$ takes an arbitrary card Y, and the partner of Y becomes the new extra element.

Thus, on his last move $\mathcal{S}$ is forced to take the extra element. After that player $\mathcal{F}$ has cards A and $A \triangle B$, player $\mathcal{S}$ has cards B and $\varnothing$, and $\mathcal{F}$ has exactly one element from every other pair. Thus the situation is the same as in the previous case with roles reversed, and $\mathcal{F}$ wins.

Finally, if $\mathcal{S}$ takes $\varnothing$ on his first move then $\mathcal{F}$ denotes any card which has not been taken yet by B and takes $A \triangle B$. After that, the same strategy as above is applicable.

Comment 1. If one wants to avoid the unusual question about the first move, one may change the formulation as follows. (The difficulty of the problem would decrease somewhat.)

A card deck consists of 1023 cards; on each card, a nonempty set of distinct decimal digits is written in such a way that no two of these sets coincide. Two players alternately take cards from the deck, one card per turn. When the deck is empty, each player checks if he can throw out one of his cards so that for each of the ten digits, he still holds an even number of cards with this digit. If one player can do this but the other one cannot, the one who can is the winner; otherwise a draw is declared.

Determine which of the players (if any) has a winning strategy.

The winner in this version is the first player. The analysis of the game from the first two paragraphs of the previous solution applies to this version as well, except for the case $C = \varnothing$ in which the result is a draw. Then the strategy for $\mathcal{S}$ in Case 1 works for $\mathcal{F}$ in this version: the sum of all his cards at the end is either A or $A \triangle B$, thus nonempty in both cases.

Comment 2. Notice that all the cards form a vector space over $\mathbb{F}_2$, with $\triangle$ the operation of addition. Due to the automorphisms of this space, all possibilities for $\mathcal{F}$'s first move except $\varnothing$ are equivalent. The same holds for the response by $\mathcal{S}$ if $\mathcal{F}$ takes the card $\varnothing$ on his first move.

Comment 3. It is not that hard to show that in the initial game, $\mathcal{F}$ has a winning move, by the idea of "strategy stealing".

Namely, assume that $\mathcal{S}$ has a winning strategy. Let us take two card decks and start two games, in which $\mathcal{S}$ will act by his strategy. In the first game, $\mathcal{F}$ takes an arbitrary card A_1; assume that $\mathcal{S}$ takes some B_1 in response. Then $\mathcal{F}$ takes the card B_1 at the second game; let the response by $\mathcal{S}$ be A_2. Then $\mathcal{F}$ takes A_2 in the first game and gets a response B_2, and so on.

This process stops at some moment when in the second game $\mathcal{S}$ takes $A_i = A_1$. At this moment the players hold the same sets of cards in both games, but with roles reversed. Now, if some cards remain in the decks, $\mathcal{F}$ takes an arbitrary card from the first deck starting a similar cycle.

At the end of the game, player $\mathcal{F}$'s cards in the first game are exactly player $\mathcal{S}$'s cards in the second game, and vice versa. Thus in one of the games $\mathcal{F}$ will win, which is impossible by our assumption.

One may notice that the strategy in Case 2 is constructed exactly in this way from the strategy in Case 1. This is possible since every response by $\mathcal{S}$ wins if $\mathcal{F}$ takes the card $\varnothing$ on his first move.

C9. There are n circles drawn on a piece of paper in such a way that any two circles intersect in two points, and no three circles pass through the same point. Turbo the snail slides along the circles in the following fashion. Initially he moves on one of the circles in clockwise direction. Turbo always keeps sliding along the current circle until he reaches an intersection with another circle. Then he continues his journey on this new circle and also changes the direction of moving, i.e. from clockwise to anticlockwise or *vice versa*.

Suppose that Turbo's path entirely covers all circles. Prove that n must be odd.

(India)

Solution 1. Replace every cross (i.e. intersection of two circles) by two small circle arcs that indicate the direction in which the snail should leave the cross (see Figure 1.1). Notice that the placement of the small arcs does not depend on the direction of moving on the curves; no matter which direction the snail is moving on the circle arcs, he will follow the same curves (see Figure 1.2). In this way we have a set of curves, that are the possible paths of the snail. Call these curves *snail orbits* or just *orbits*. Every snail orbit is a simple closed curve that has no intersection with any other orbit.

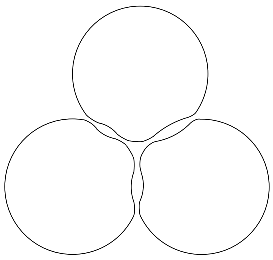


Figure 1.1

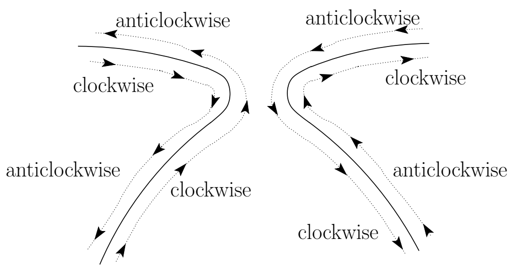


Figure 1.2

We prove the following, more general statement.

> $(*)$ *In any configuration of n circles such that no two of them are tangent, the number of snail orbits has the same parity as the number n.* (Note that it is not assumed that all circle pairs intersect.)

This immediately solves the problem.

Let us introduce the following operation that will be called *flipping a cross*. At a cross, remove the two small arcs of the orbits, and replace them by the other two arcs. Hence, when the snail arrives at a flipped cross, he will continue on the other circle as before, but he will preserve the orientation in which he goes along the circle arcs (see Figure 2).

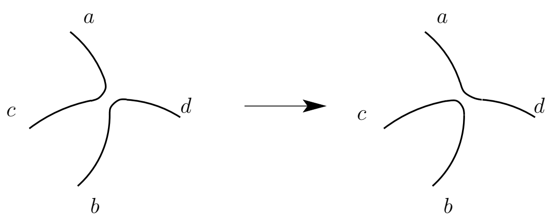


Figure 2

Consider what happens to the number of orbits when a cross is flipped. Denote by a, b, c, and d the four arcs that meet at the cross such that a and b belong to the same circle. Before the flipping a and b were connected to c and d, respectively, and after the flipping a and b are connected to d and c, respectively.

The orbits passing through the cross are closed curves, so each of the arcs a, b, c, and d is connected to another one by orbits outside the cross. We distinguish three cases.

Case 1: a is connected to b and c is connected to d by the orbits outside the cross (see Figure 3.1).

We show that this case is impossible. Remove the two small arcs at the cross, connect a to b, and connect c to d at the cross. Let γ be the new closed curve containing a and b, and let δ be the new curve that connects c and d. These two curves intersect at the cross. So one of c and d is inside γ and the other one is outside γ. Then the two closed curves have to meet at least one more time, but this is a contradiction, since no orbit can intersect itself.

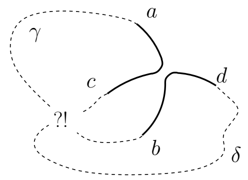


Figure 3.1

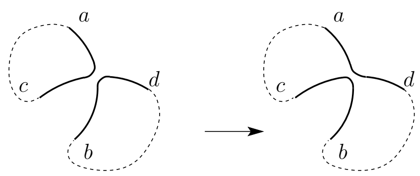


Figure 3.2

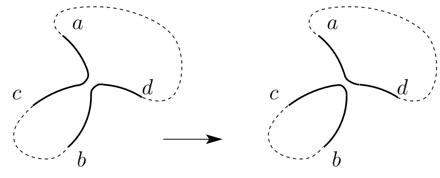


Figure 3.3

Case 2: a is connected to c and b is connected to d (see Figure 3.2).

Before the flipping a and c belong to one orbit and b and d belong to another orbit. Flipping the cross merges the two orbits into a single orbit. Hence, the number of orbits decreases by 1.

Case 3: a is connected to d and b is connected to c (see Figure 3.3).

Before the flipping the arcs a, b, c, and d belong to a single orbit. Flipping the cross splits that orbit in two. The number of orbits increases by 1.

As can be seen, every flipping decreases or increases the number of orbits by one, thus changes its parity.

Now flip every cross, one by one. Since every pair of circles has 0 or 2 intersections, the number of crosses is even. Therefore, when all crosses have been flipped, the original parity of the number of orbits is restored. So it is sufficient to prove $(*)$ for the new configuration, where all crosses are flipped. Of course also in this new configuration the (modified) orbits are simple closed curves not intersecting each other.

Orient the orbits in such a way that the snail always moves anticlockwise along the circle arcs. Figure 4 shows the same circles as in Figure 1 after flipping all crosses and adding orientation. (Note that this orientation may be different from the orientation of the orbit as a planar curve; the orientation of every orbit may be negative as well as positive, like the middle orbit in Figure 4.) If the snail moves around an orbit, the total angle change in his moving direction, the *total curvature*, is either $+2\pi$ or -2π, depending on the orientation of the orbit. Let P and N be the number of orbits with positive and negative orientation, respectively. Then the total curvature of all orbits is $(P-N)\cdot 2\pi$.

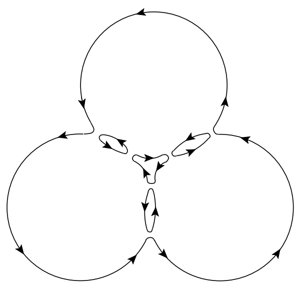


Figure 4

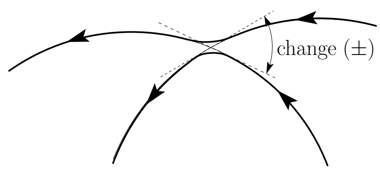


Figure 5

Double-count the total curvature of all orbits. Along every circle the total curvature is 2π. At every cross, the two turnings make two changes with some angles having the same absolute value but opposite signs, as depicted in Figure 5. So the changes in the direction at the crosses cancel out. Hence, the total curvature is $n \cdot 2\pi$.

Now we have $(P-N)\cdot 2\pi = n\cdot 2\pi$, so $P-N=n$. The number of (modified) orbits is $P+N$, that has a same parity as $P-N=n$.

Solution 2. We present a different proof of $(*)$.

We perform a sequence of small modification steps on the configuration of the circles in such a way that at the end they have no intersection at all (see Figure 6.1). We use two kinds of local changes to the structure of the orbits (see Figure 6.2):

- *Type-1 step:* An arc of a circle is moved over an arc of another circle; such a step creates or removes two intersections.

- *Type-2 step:* An arc of a circle is moved through the intersection of two other circles.

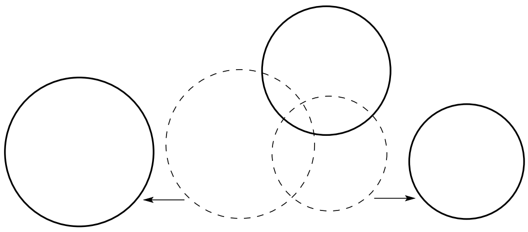


Figure 6.1

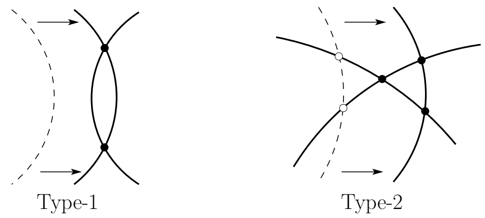


Figure 6.2

We assume that in every step only one circle is moved, and that this circle is moved over at most one arc or intersection point of other circles.

We will show that the parity of the number of orbits does not change in any step. As every circle becomes a separate orbit at the end of the procedure, this fact proves $(*)$.

Consider what happens to the number of orbits when a Type-1 step is performed. The two intersection points are created or removed in a small neighbourhood. Denote some points of the two circles where they enter or leave this neighbourhood by a, b, c, and d in this order around the neighbourhood; let a and b belong to one circle and let c and d belong to the other circle. The two circle arcs may have the same or opposite orientations. Moreover, the four end-points of the two arcs are connected by the other parts of the orbits. This can happen in two ways without intersection: either a is connected to d and b is connected to c, or a is connected to b and c is connected to d. Altogether we have four cases, as shown in Figure 7.

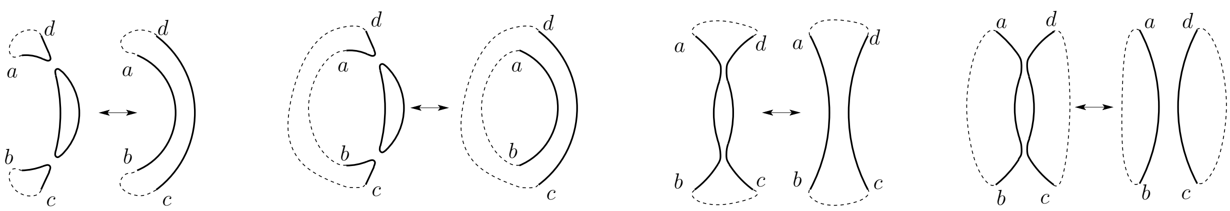


Figure 7

We can see that the number of orbits is changed by -2 or $+2$ in the leftmost case when the arcs have the same orientation, a is connected to d, and b is connected to c. In the other three cases the number of orbits is not changed. Hence, Type-1 steps do not change the parity of the number of orbits.

Now consider a Type-2 step. The three circles enclose a small, triangular region; by the step, this triangle is replaced by another triangle. Again, the modification of the orbits is done in some small neighbourhood; the structure does not change outside. Each side of the triangle shaped region can be convex or concave; the number of concave sides can be 0, 1, 2 or 3, so there are 4 possible arrangements of the orbits inside the neighbourhood, as shown in Figure 8.

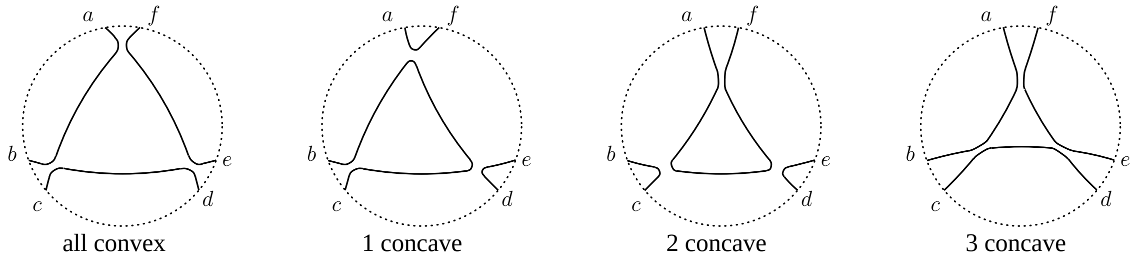


Figure 8

Denote the points where the three circles enter or leave the neighbourhood by a, b, c, d, e, and f in this order around the neighbourhood. As can be seen in Figure 8, there are only two essentially different cases; either a, c, e are connected to b, d, f, respectively, or a, c, e are connected to f, b, d, respectively. The step either preserves the set of connections or switches to the other arrangement. Obviously, in the earlier case the number of orbits is not changed; therefore we have to consider only the latter case.

The points a, b, c, d, e, and f are connected by the orbits outside, without intersection. If a was connected to c, say, then this orbit would isolate b, so this is impossible. Hence, each of a, b, c, d, e and f must be connected either to one of its neighbours or to the opposite point. If say a is connected to d, then this orbit separates b and c from e and f, therefore b must be connected to c and e must be connected to f. Altogether there are only two cases and their reverses: either each point is connected to one of its neighbours or two opposite points are connected and the the remaining neigh boring pairs are connected to each other. See Figure 9.

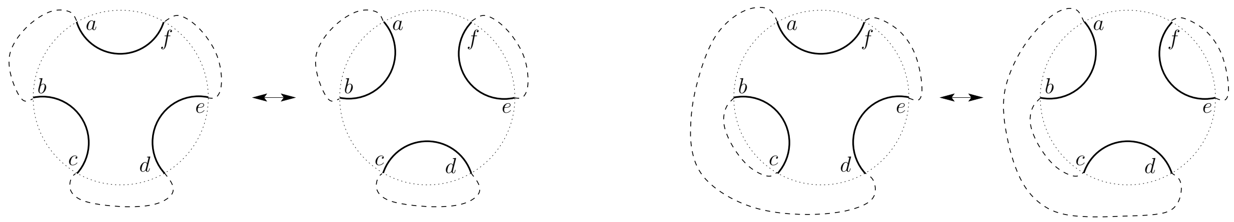


Figure 9

We can see that if only neighbouring points are connected, then the number of orbits is changed by $+2$ or -2. If two opposite points are connected (a and d in the figure), then the orbits are re-arranged, but their number is unchanged. Hence, Type-2 steps also preserve the parity. This completes the proof of $(*)$.

Solution 3. Like in the previous solutions, we do not need all circle pairs to intersect but we assume that the circles form a connected set. Denote by $\mathcal{C}$ and $\mathcal{P}$ the sets of circles and their intersection points, respectively.

The circles divide the plane into several simply connected, bounded regions and one unbounded region. Denote the set of these regions by $\mathcal{R}$. We say that an intersection point or a region is *odd* or *even* if it is contained inside an odd or even number of circles, respectively. Let $\mathcal{P}_{\text{odd}}$ and $\mathcal{R}_{\text{odd}}$ be the sets of odd intersection points and odd regions, respectively.

Claim.

$$|\mathcal{R}_{\text{odd}}| - |\mathcal{P}_{\text{odd}}| \equiv n \pmod 2. \tag{1}$$

Proof. For each circle $c \in \mathcal{C}$, denote by R_c, P_c, and X_c the number of regions inside c, the number of intersection points inside c, and the number of circles intersecting c, respectively. The circles divide each other into several arcs; denote by A_c the number of such arcs inside c. By double counting the regions and intersection points inside the circles we get

$$|\mathcal{R}_{\text{odd}}| \equiv \sum_{c \in \mathcal{C}} R_c \pmod 2 \quad \text{and} \quad |\mathcal{P}_{\text{odd}}| \equiv \sum_{c \in \mathcal{C}} P_c \pmod 2.$$

For each circle c, apply Euler's polyhedron theorem to the (simply connected) regions in c. There are $2X_c$ intersection points on c; they divide the circle into $2X_c$ arcs. The polyhedron theorem yields $(R_c+1)+(P_c+2X_c)=(A_c+2X_c)+2$, considering the exterior of c as a single region. Therefore,

$$R_c + P_c = A_c + 1. \tag{2}$$

Moreover, we have four arcs starting from every interior points inside c and a single arc starting into the interior from each intersection point on the circle. By double-counting the end-points of the interior arcs we get $2A_c = 4P_c + 2X_c$, so

$$A_c = 2P_c + X_c. \tag{3}$$

The relations (2) and (3) together yield

$$R_c - P_c = X_c + 1. \tag{4}$$

By summing up (4) for all circles we obtain

$$\sum_{c\in\mathcal{C}} R_c - \sum_{c\in\mathcal{C}} P_c = \sum_{c\in\mathcal{C}} X_c + |\mathcal{C}|,$$

which yields

$$|\mathcal{R}_{\text{odd}}| - |\mathcal{P}_{\text{odd}}| \equiv \sum_{c\in\mathcal{C}} X_c + n \pmod 2. \tag{5}$$

Notice that in $\sum\limits_{c\in\mathcal{C}} X_c$ each intersecting circle pair is counted twice, i.e., for both circles in the pair, so

$$\sum_{c\in\mathcal{C}} X_c \equiv 0 \pmod 2,$$

which finishes the proof of the Claim. □

Now insert the same small arcs at the intersections as in the first solution, and suppose that there is a single snail orbit b.

First we show that the odd regions are inside the curve b, while the even regions are outside. Take a region $r \in \mathcal{R}$ and a point x in its interior, and draw a ray y, starting from x, that does not pass through any intersection point of the circles and is neither tangent to any of the circles. As is well-known, x is inside the curve b if and only if y intersects b an odd number of times (see Figure 10). Notice that if an arbitrary circle c contains x in its interior, then c intersects y at a single point; otherwise, if x is outside c, then c has 2 or 0 intersections with y. Therefore, y intersects b an odd number of times if and only if x is contained in an odd number of circles, so if and only if r is odd.

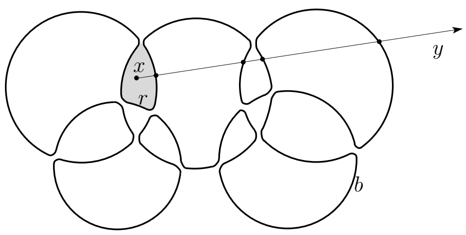


Figure 10

Now consider an intersection point p of two circles c_1 and c_2 and a small neighbourhood around p. Suppose that p is contained inside k circles.

We have four regions that meet at p. Let r_1 be the region that lies outside both c_1 and c_2, let r_2 be the region that lies inside both c_1 and c_2, and let r_3 and r_4 be the two remaining regions, each lying inside exactly one of c_1 and c_2. The region r_1 is contained inside the same k circles as p; the region r_2 is contained also by c_1 and c_2, so by $k+2$ circles in total; each of the regions r_3 and r_4 is contained inside $k+1$ circles. After the small arcs have been inserted at p, the regions r_1 and r_2 get connected, and the regions r_3 and r_4 remain separated at p (see Figure 11). If p is an odd point, then r_1 and r_2 are odd, so two odd regions are connected at p. Otherwise, if p is even, then we have two even regions connected at p.

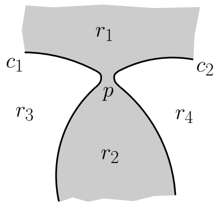


Figure 11

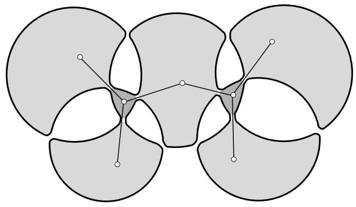


Figure 12

Consider the system of odd regions and their connections at the odd points as a graph. In this graph the odd regions are the vertices, and each odd point establishes an edge that connects two vertices (see Figure 12). As b is a single closed curve, this graph is connected and contains no cycle, so the graph is a tree. Then the number of vertices must be by one greater than the number of edges, so

$$|\mathcal{R}_{\mathrm{odd}}| - |\mathcal{P}_{\mathrm{odd}}| = 1. \tag{9}$$

The relations (1) and (9) together prove that n must be odd.

Comment. For every odd n there exists at least one configuration of n circles with a single snail orbit. Figure 13 shows a possible configuration with 5 circles. In general, if a circle is rotated by $k \cdot \frac{360^\circ}{n}$ ($k = 1, 2, \ldots, n-1$) around an interior point other than the centre, the circle and its rotated copies together provide a single snail orbit.

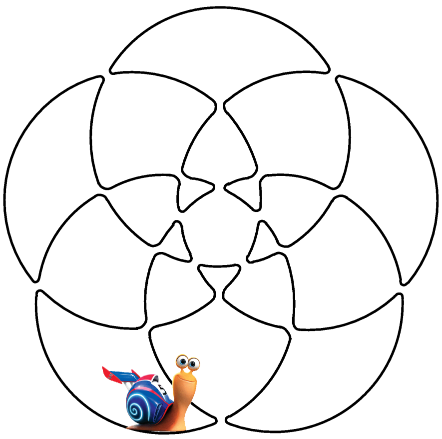


Figure 13

Geometry

G1. The points P and Q are chosen on the side BC of an acute-angled triangle ABC so that $\angle PAB = \angle ACB$ and $\angle QAC = \angle CBA$. The points M and N are taken on the rays AP and AQ, respectively, so that $AP = PM$ and $AQ = QN$. Prove that the lines BM and CN intersect on the circumcircle of the triangle ABC.

(Georgia)

Solution 1. Denote by S the intersection point of the lines BM and CN. Let moreover $\beta = \angle QAC = \angle CBA$ and $\gamma = \angle PAB = \angle ACB$. From these equalities it follows that the triangles ABP and CAQ are similar (see Figure 1). Therefore we obtain

$$\frac{BP}{PM} = \frac{BP}{PA} = \frac{AQ}{QC} = \frac{NQ}{QC} .$$

Moreover,

$$\angle BPM = \beta + \gamma = \angle CQN .$$

Hence the triangles BPM and NQC are similar. This gives $\angle BMP = \angle NCQ$, so the triangles BPM and BSC are also similar. Thus we get

$$\angle CSB = \angle BPM = \beta + \gamma = 180^\circ - \angle BAC ,$$

which completes the solution.

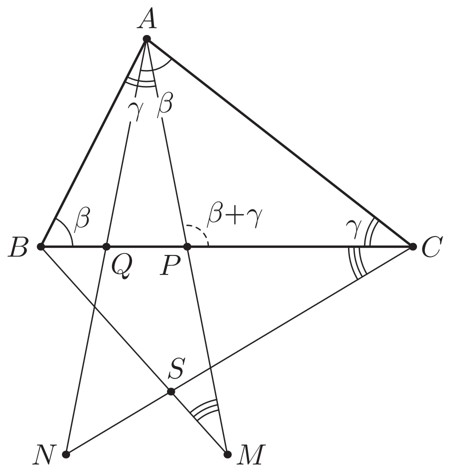


Figure 1

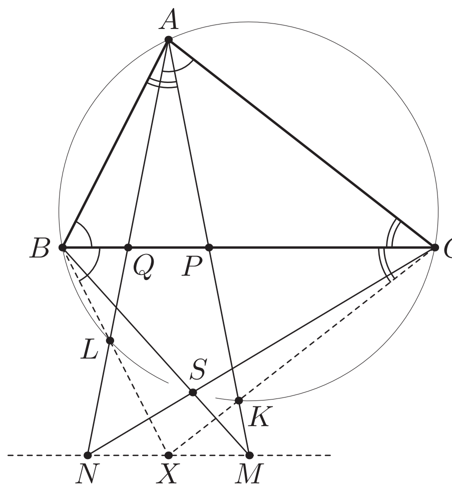


Figure 2

Solution 2. As in the previous solution, denote by S the intersection point of the lines BM and NC. Let moreover the circumcircle of the triangle ABC intersect the lines AP and AQ again at K and L, respectively (see Figure 2).

Note that $\angle LBC = \angle LAC = \angle CBA$ and similarly $\angle KCB = \angle KAB = \angle BCA$. It implies that the lines BL and CK meet at a point X, being symmetric to the point A with respect to the line BC. Since $AP = PM$ and $AQ = QN$, it follows that X lies on the line MN. Therefore, using Pascal's theorem for the hexagon $ALBSCK$, we infer that S lies on the circumcircle of the triangle ABC, which finishes the proof.

Comment. Both solutions can be modified to obtain a more general result, with the equalities

$$AP = PM \quad \text{and} \quad AQ = QN$$

replaced by

$$\frac{AP}{PM} = \frac{QN}{AQ} .$$

G2. Let ABC be a triangle. The points K, L, and M lie on the segments BC, CA, and AB, respectively, such that the lines AK, BL, and CM intersect in a common point. Prove that it is possible to choose two of the triangles ALM, BMK, and CKL whose inradii sum up to at least the inradius of the triangle ABC.

(Estonia)

Solution. Denote

$$a=\frac{BK}{KC}, \qquad b=\frac{CL}{LA}, \qquad c=\frac{AM}{MB}.$$

By CEVA's theorem, $abc=1$, so we may, without loss of generality, assume that $a \geqslant 1$. Then at least one of the numbers b or c is not greater than 1. Therefore at least one of the pairs (a,b), (b,c) has its first component not less than 1 and the second one not greater than 1. Without loss of generality, assume that $1 \leqslant a$ and $b \leqslant 1$.

Therefore, we obtain $bc \leqslant 1$ and $1 \leqslant ca$, or equivalently

$$\frac{AM}{MB} \leqslant \frac{LA}{CL} \qquad \text{and} \qquad \frac{MB}{AM} \leqslant \frac{BK}{KC}.$$

The first inequality implies that the line passing through M and parallel to BC intersects the *segment* AL at a point X (see Figure 1). Therefore the inradius of the triangle ALM is not less than the inradius r_1 of triangle AMX.

Similarly, the line passing through M and parallel to AC intersects the *segment* BK at a point Y, so the inradius of the triangle BMK is not less than the inradius r_2 of the triangle BMY. Thus, to complete our solution, it is enough to show that $r_1+r_2 \geqslant r$, where r is the inradius of the triangle ABC. We prove that in fact $r_1+r_2=r$.

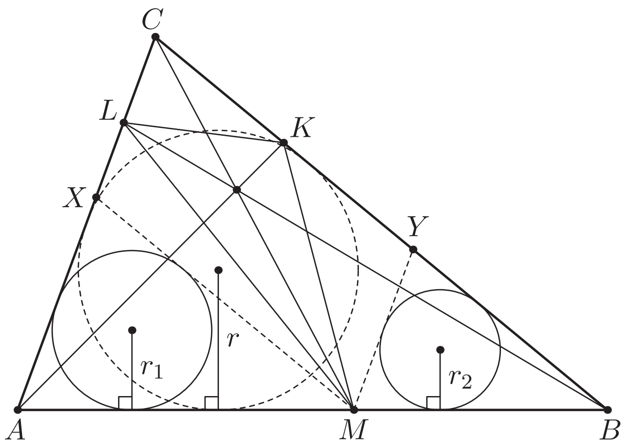


Figure 1

Since $MX \parallel BC$, the dilation with centre A that takes M to B takes the incircle of the triangle AMX to the incircle of the triangle ABC. Therefore

$$\frac{r_1}{r}=\frac{AM}{AB}, \qquad \text{and similarly} \quad \frac{r_2}{r}=\frac{MB}{AB}.$$

Adding these equalities gives $r_1+r_2=r$, as required.

Comment. Alternatively, one can use DESARGUES' theorem instead of CEVA's theorem, as follows: The lines AB, BC, CA dissect the plane into seven regions. One of them is bounded, and amongst the other six, three are two-sided and three are three-sided. Now define the points $P=BC\cap LM$, $Q=CA\cap MK$, and $R=AB\cap KL$ (in the projective plane). By DESARGUES' theorem, the points P, Q, R lie on a common line ℓ. This line intersects only unbounded regions. If we now assume (without loss of generality) that P, Q and R lie on ℓ in that order, then one of the segments PQ or QR lies inside a two-sided region. If, for example, this segment is PQ, then the triangles ALM and BMK will satisfy the statement of the problem for the same reason.

G3. Let Ω and O be the circumcircle and the circumcentre of an acute-angled triangle ABC with $AB > BC$. The angle bisector of $\angle ABC$ intersects Ω at $M \neq B$. Let Γ be the circle with diameter BM. The angle bisectors of $\angle AOB$ and $\angle BOC$ intersect Γ at points P and Q, respectively. The point R is chosen on the line PQ so that $BR = MR$. Prove that $BR \parallel AC$. (Here we always assume that an angle bisector is a ray.)

(Russia)

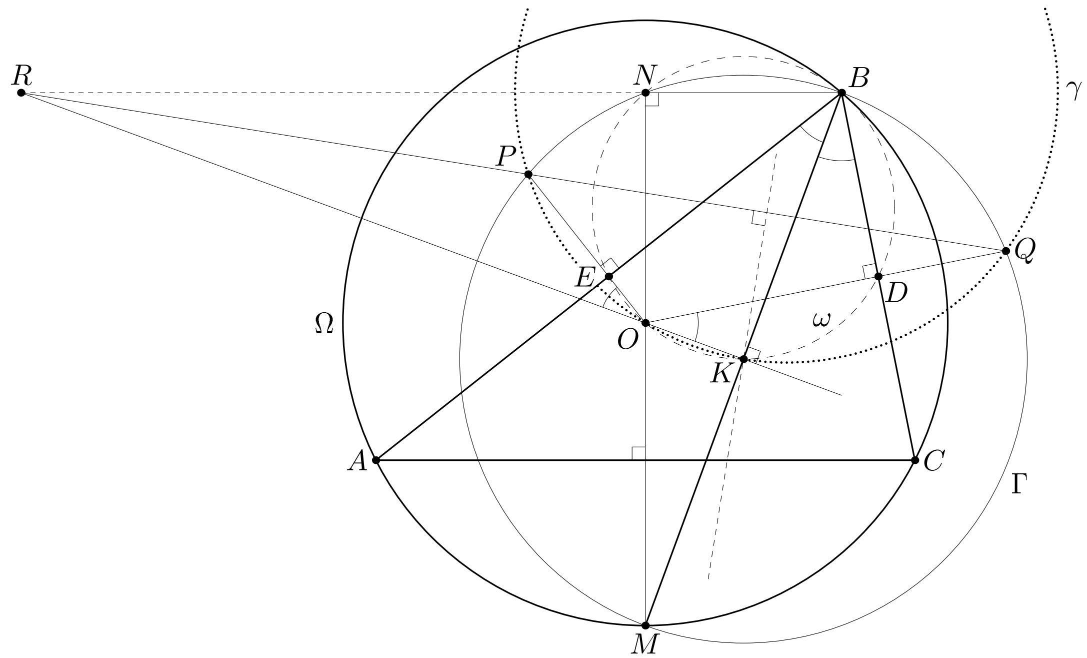
Solution. Let K be the midpoint of BM, i.e., the centre of Γ. Notice that $AB \neq BC$ implies $K \neq O$. Clearly, the lines OM and OK are the perpendicular bisectors of AC and BM, respectively. Therefore, R is the intersection point of PQ and OK.

Let N be the second point of intersection of Γ with the line OM. Since BM is a diameter of Γ, the lines BN and AC are both perpendicular to OM. Hence $BN \parallel AC$, and it suffices to prove that BN passes through R. Our plan for doing this is to interpret the lines BN, OK, and PQ as the radical axes of three appropriate circles.

Let ω be the circle with diameter BO. Since $\angle BNO = \angle BKO = 90^\circ$, the points N and K lie on ω.

Next we show that the points O, K, P, and Q are concyclic. To this end, let D and E be the midpoints of BC and AB, respectively. Clearly, D and E lie on the rays OQ and OP, respectively. By our assumptions about the triangle ABC, the points B, E, O, K, and D lie in this order on ω. It follows that $\angle EOR = \angle EBK = \angle KBD = \angle KOD$, so the line KO externally bisects the angle POQ. Since the point K is the centre of Γ, it also lies on the perpendicular bisector of PQ. So K coincides with the midpoint of the arc POQ of the circumcircle γ of triangle POQ.

Thus the lines OK, BN, and PQ are pairwise radical axes of the circles ω, γ, and Γ. Hence they are concurrent at R, as required.



G4. Consider a fixed circle Γ with three fixed points A, B, and C on it. Also, let us fix a real number $\lambda \in (0,1)$. For a variable point $P \notin \{A, B, C\}$ on Γ, let M be the point on the segment CP such that $CM = \lambda \cdot CP$. Let Q be the second point of intersection of the circumcircles of the triangles AMP and BMC. Prove that as P varies, the point Q lies on a fixed circle.

(United Kingdom)

Solution 1. Throughout the solution, we denote by $\measuredangle(a, b)$ the directed angle between the lines a and b.

Let D be the point on the segment AB such that $BD = \lambda \cdot BA$. We will show that either $Q = D$, or $\measuredangle(DQ, QB) = \measuredangle(AB, BC)$; this would mean that the point Q varies over the constant circle through D tangent to BC at B, as required.

Denote the circumcircles of the triangles AMP and BMC by ω_A and ω_B, respectively. The lines AP, BC, and MQ are pairwise radical axes of the circles Γ, ω_A, and ω_B, thus either they are parallel, or they share a common point X.

Assume that these lines are parallel (see Figure 1). Then the segments AP, QM, and BC have a common perpendicular bisector; the reflection in this bisector maps the segment CP to BA, and maps M to Q. Therefore, in this case Q lies on AB, and $BQ/AB = CM/CP = BD/AB$; so we have $Q = D$.

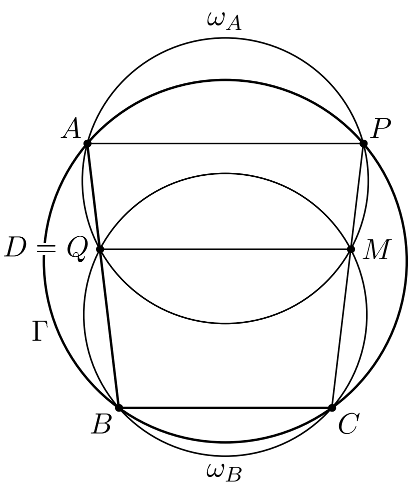


Figure 1

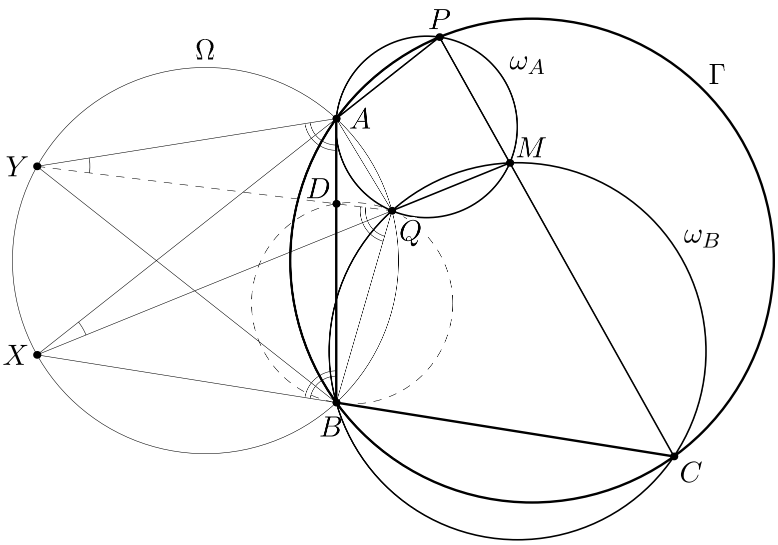


Figure 2

Now assume that the lines AP, QM, and BC are concurrent at some point X (see Figure 2). Notice that the points A, B, Q, and X lie on a common circle Ω by Miquel's theorem applied to the triangle XPC. Let us denote by Y the symmetric image of X about the perpendicular bisector of AB. Clearly, Y lies on Ω, and the triangles YAB and $\triangle XBA$ are congruent. Moreover, the triangle XPC is similar to the triangle XBA, so it is also similar to the triangle YAB.

Next, the points D and M correspond to each other in similar triangles YAB and XPC, since $BD/BA = CM/CP = \lambda$. Moreover, the triangles YAB and XPC are equi-oriented, so $\measuredangle(MX, XP) = \measuredangle(DY, YA)$. On the other hand, since the points A, Q, X, and Y lie on Ω, we have $\measuredangle(QY, YA) = \measuredangle(MX, XP)$. Therefore, $\measuredangle(QY, YA) = \measuredangle(DY, YA)$, so the points Y, D, and Q are collinear.

Finally, we have $\measuredangle(DQ, QB) = \measuredangle(YQ, QB) = \measuredangle(YA, AB) = \measuredangle(AB, BX) = \measuredangle(AB, BC)$, as desired.

Comment. In the original proposal, λ was supposed to be an arbitrary real number distinct from 0 and 1, and the point M was defined by $\overrightarrow{CM} = \lambda \cdot \overrightarrow{CP}$. The Problem Selection Committee decided to add the restriction $\lambda \in (0,1)$ in order to avoid a large case distinction.

Solution 2. As in the previous solution, we introduce the radical centre $X = AP \cap BC \cap MQ$ of the circles ω_A, ω_B, and Γ. Next, we also notice that the points A, Q, B, and X lie on a common circle Ω.

If the point P lies on the arc BAC of Γ, then the point X is outside Γ, thus the point Q belongs to the ray XM, and therefore the points P, A, and Q lie on the same side of BC. Otherwise, if P lies on the arc BC not containing A, then X lies inside Γ, so M and Q lie on different sides of BC; thus again Q and A lie on the same side of BC. So, in each case the points Q and A lie on the same side of BC.

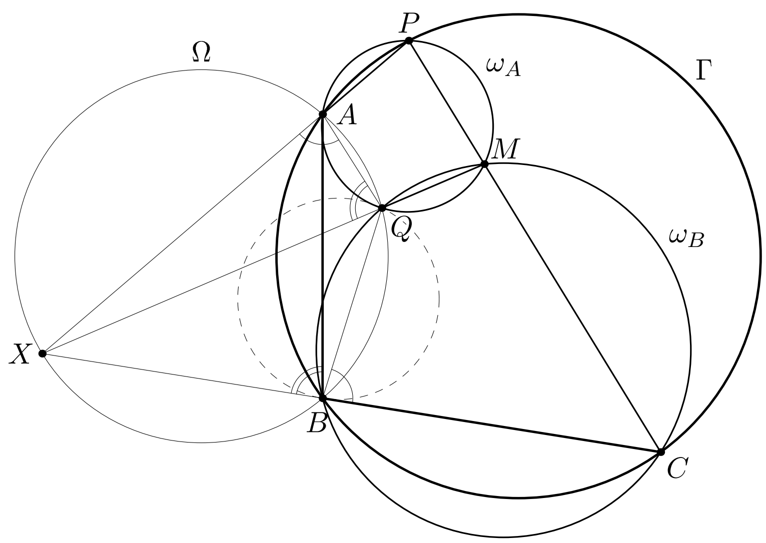


Figure 3

Now we prove that the ratio

$$\frac{QB}{\sin \angle QBC} = \frac{QB}{QX} \cdot \frac{QX}{\sin \angle QBX}$$

is constant. Since the points A, Q, B, and X are concyclic, we have

$$\frac{QX}{\sin \angle QBX} = \frac{AX}{\sin \angle ABC}.$$

Next, since the points B, Q, M, and C are concyclic, the triangles XBQ and XMC are similar, so

$$\frac{QB}{QX} = \frac{CM}{CX} = \lambda \cdot \frac{CP}{CX}.$$

Analogously, the triangles XCP and XAB are also similar, so

$$\frac{CP}{CX} = \frac{AB}{AX}.$$

Therefore, we obtain

$$\frac{QB}{\sin \angle QBC} = \lambda \cdot \frac{AB}{AX} \cdot \frac{AX}{\sin \angle ABC} = \lambda \cdot \frac{AB}{\sin \angle ABC},$$

so this ratio is indeed constant. Thus the circle passing through Q and tangent to BC at B is also constant, and Q varies over this fixed circle.

Comment. It is not hard to guess that the desired circle should be tangent to BC at B. Indeed, the second paragraph of this solution shows that this circle lies on one side of BC; on the other hand, in the limit case $P = B$, the point Q also coincides with B.

Solution 3. Let us perform an inversion centred at C. Denote by X' the image of a point X under this inversion.

The circle Γ maps to the line Γ' passing through the constant points A' and B', and containing the variable point P'. By the problem condition, the point M varies over the circle γ which is the homothetic image of Γ with centre C and coefficient λ. Thus M' varies over the constant line $\gamma' \parallel A'B'$ which is the homothetic image of $A'B'$ with centre C and coefficient $1/\lambda$, and $M = \gamma' \cap CP'$. Next, the circumcircles ω_A and ω_B of the triangles AMP and BMC map to the circumcircle ω'_A of the triangle $A'M'P'$ and to the line $B'M'$, respectively; the point Q thus maps to the second point of intersection of $B'M'$ with ω'_A (see Figure 4).

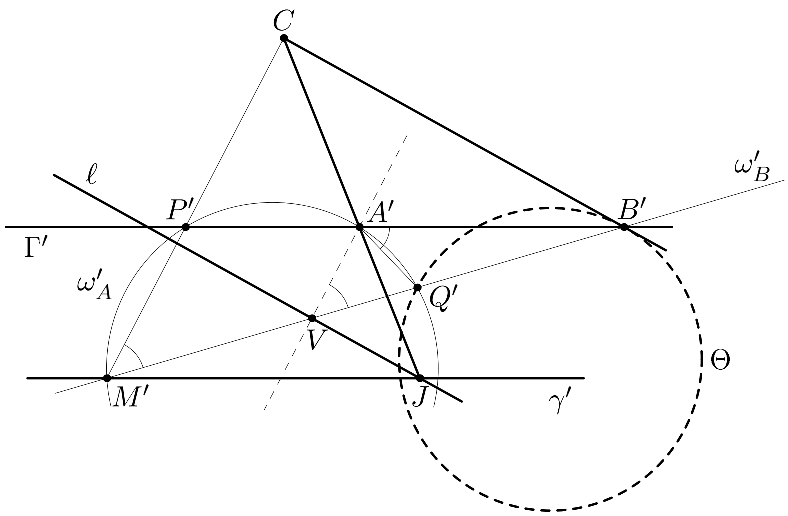


Figure 4

Let J be the (constant) common point of the lines γ' and CA', and let ℓ be the (constant) line through J parallel to CB'. Let V be the common point of the lines ℓ and $B'M'$. Applying Pappus' theorem to the triples (C, J, A') and (V, B', M') we get that the points $CB' \cap JV$, $JM' \cap A'B'$, and $CM' \cap A'V$ are collinear. The first two of these points are ideal, hence so is the third, which means that $CM' \parallel A'V$.

Now we have $\measuredangle(Q'A', A'P') = \measuredangle(Q'M', M'P') = \angle(VM', A'V)$, which means that the triangles $B'Q'A'$ and $B'A'V$ are similar, and $(B'A')^2 = B'Q' \cdot B'V$. Thus Q' is the image of V under the second (fixed) inversion with centre B' and radius $B'A'$. Since V varies over the constant line ℓ, Q' varies over some constant circle Θ. Thus, applying the first inversion back we get that Q also varies over some fixed circle.

One should notice that this last circle is not a line; otherwise Θ would contain C, and thus ℓ would contain the image of C under the second inversion. This is impossible, since $CB' \parallel \ell$.

G5. Let $ABCD$ be a convex quadrilateral with $\angle B = \angle D = 90^\circ$. Point H is the foot of the perpendicular from A to BD. The points S and T are chosen on the sides AB and AD, respectively, in such a way that H lies inside triangle SCT and

$$\angle SHC - \angle BSC = 90^\circ, \quad \angle THC - \angle DTC = 90^\circ\,.$$

Prove that the circumcircle of triangle SHT is tangent to the line BD.

(Iran)

Solution. Let the line passing through C and perpendicular to the line SC intersect the line AB at Q (see Figure 1). Then

$$\angle SQC = 90^\circ - \angle BSC = 180^\circ - \angle SHC\,,$$

which implies that the points C, H, S, and Q lie on a common circle. Moreover, since SQ is a diameter of this circle, we infer that the circumcentre K of triangle SHC lies on the line AB. Similarly, we prove that the circumcentre L of triangle CHT lies on the line AD.

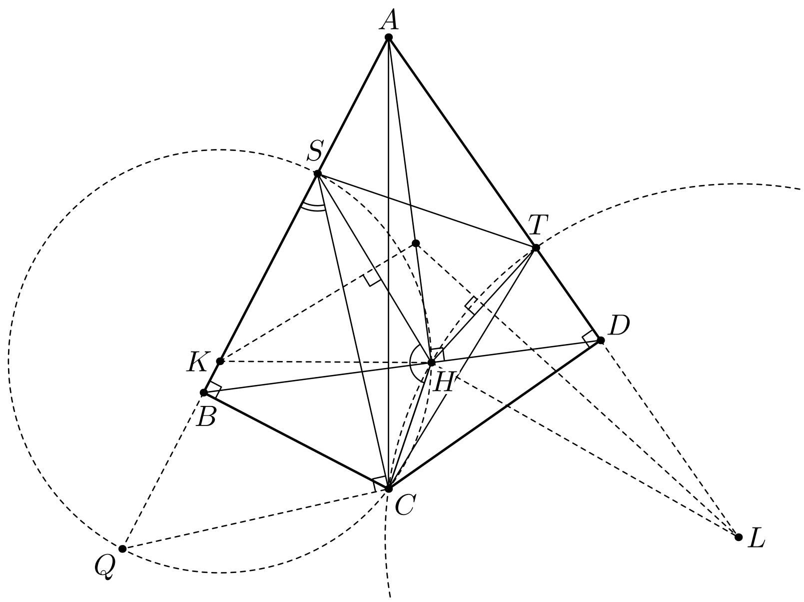


Figure 1

In order to prove that the circumcircle of triangle SHT is tangent to BD, it suffices to show that the perpendicular bisectors of HS and HT intersect on the line AH. However, these two perpendicular bisectors coincide with the angle bisectors of angles AKH and ALH. Therefore, in order to complete the solution, it is enough (by the bisector theorem) to show that

$$\frac{AK}{KH} = \frac{AL}{LH}\,. \tag{1}$$

We present two proofs of this equality.

First proof. Let the lines KL and HC intersect at M (see Figure 2). Since $KH = KC$ and $LH = LC$, the points H and C are symmetric to each other with respect to the line KL. Therefore M is the midpoint of HC. Denote by O the circumcentre of quadrilateral $ABCD$. Then O is the midpoint of AC. Therefore we have $OM \parallel AH$ and hence $OM \perp BD$. This together with the equality $OB = OD$ implies that OM is the perpendicular bisector of BD and therefore $BM = DM$.

Since $CM \perp KL$, the points B, C, M, and K lie on a common circle with diameter KC. Similarly, the points L, C, M, and D lie on a circle with diameter LC. Thus, using the sine law, we obtain

$$\frac{AK}{AL} = \frac{\sin\angle ALK}{\sin\angle AKL} = \frac{DM}{CL}\cdot\frac{CK}{BM} = \frac{CK}{CL} = \frac{KH}{LH}\,,$$

which finishes the proof of (1).

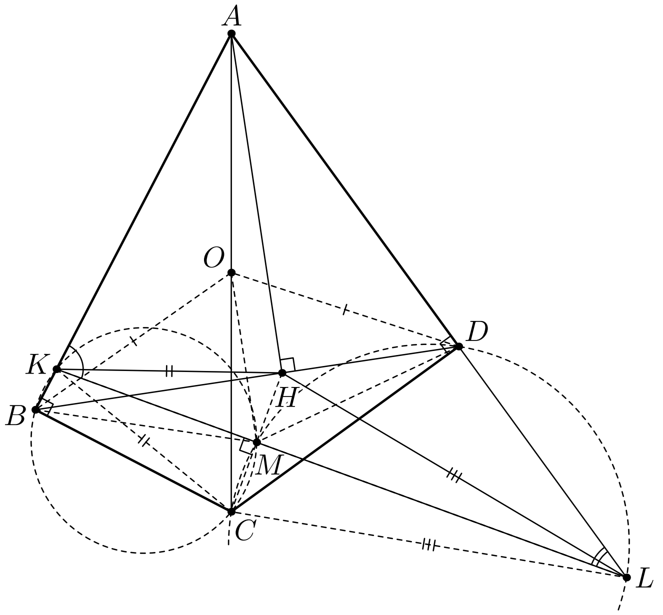


Figure 2

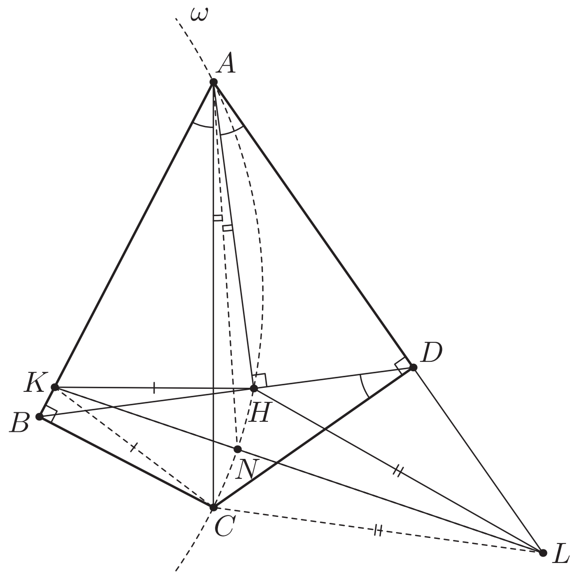


Figure 3

Second proof. If the points A, H, and C are collinear, then $AK = AL$ and $KH = LH$, so the equality (1) follows. Assume therefore that the points A, H, and C do not lie in a line and consider the circle ω passing through them (see Figure 3). Since the quadrilateral $ABCD$ is cyclic,

$$\angle BAC = \angle BDC = 90^\circ - \angle ADH = \angle HAD\,.$$

Let $N \neq A$ be the intersection point of the circle ω and the angle bisector of $\angle CAH$. Then AN is also the angle bisector of $\angle BAD$. Since H and C are symmetric to each other with respect to the line KL and $HN = NC$, it follows that both N and the centre of ω lie on the line KL. This means that the circle ω is an Apollonius circle of the points K and L. This immediately yields (1).

Comment. Either proof can be used to obtain the following generalised result:

Let $ABCD$ be a convex quadrilateral and let H be a point in its interior with $\angle BAC = \angle DAH$. The points S and T are chosen on the sides AB and AD, respectively, in such a way that H lies inside triangle SCT and

$$\angle SHC - \angle BSC = 90^\circ, \quad \angle THC - \angle DTC = 90^\circ\,.$$

Then the circumcentre of triangle SHT lies on the line AH (and moreover the circumcentre of triangle SCT lies on AC).

G6. Let ABC be a fixed acute-angled triangle. Consider some points E and F lying on the sides AC and AB, respectively, and let M be the midpoint of EF. Let the perpendicular bisector of EF intersect the line BC at K, and let the perpendicular bisector of MK intersect the lines AC and AB at S and T, respectively. We call the pair (E,F) *interesting*, if the quadrilateral $KSAT$ is cyclic.

Suppose that the pairs (E_1,F_1) and (E_2,F_2) are interesting. Prove that

$$\frac{E_1E_2}{AB}=\frac{F_1F_2}{AC}\,.$$

(Iran)

Solution 1. For any interesting pair (E,F), we will say that the corresponding triangle EFK is also *interesting*.

Let EFK be an interesting triangle. Firstly, we prove that $\angle KEF=\angle KFE=\angle A$, which also means that the circumcircle ω_1 of the triangle AEF is tangent to the lines KE and KF.

Denote by ω the circle passing through the points K, S, A, and T. Let the line AM intersect the line ST and the circle ω (for the second time) at N and L, respectively (see Figure 1).

Since $EF\parallel TS$ and M is the midpoint of EF, N is the midpoint of ST. Moreover, since K and M are symmetric to each other with respect to the line ST, we have $\angle KNS=\angle MNS=\angle LNT$. Thus the points K and L are symmetric to each other with respect to the perpendicular bisector of ST. Therefore $KL\parallel ST$.

Let G be the point symmetric to K with respect to N. Then G lies on the line EF, and we may assume that it lies on the ray MF. One has

$$\angle KGE=\angle KNS=\angle SNM=\angle KLA=180^\circ-\angle KSA$$

(if $K=L$, then the angle KLA is understood to be the angle between AL and the tangent to ω at L). This means that the points K, G, E, and S are concyclic. Now, since $KSGT$ is a parallelogram, we obtain $\angle KEF=\angle KSG=180^\circ-\angle TKS=\angle A$. Since $KE=KF$, we also have $\angle KFE=\angle KEF=\angle A$.

After having proved this fact, one may finish the solution by different methods.

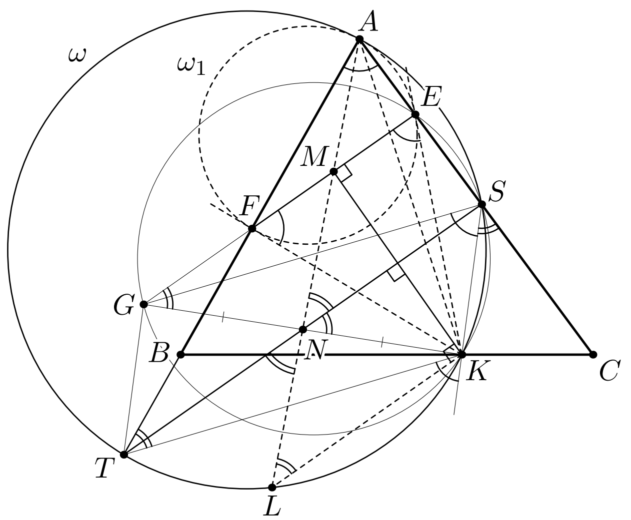


Figure 1

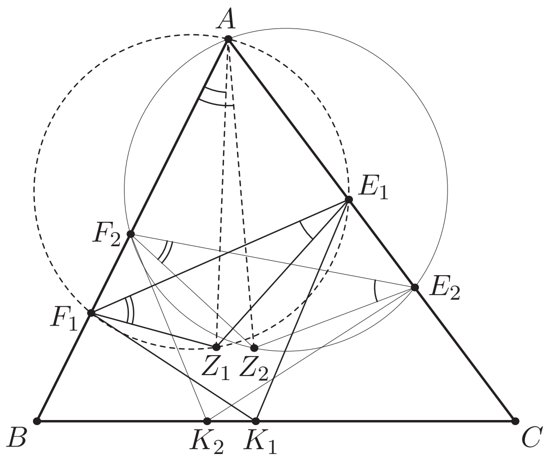


Figure 2

First method. We have just proved that all interesting triangles are similar to each other. This allows us to use the following lemma.

Lemma. Let ABC be an arbitrary triangle. Choose two points E_1 and E_2 on the side AC, two points F_1 and F_2 on the side AB, and two points K_1 and K_2 on the side BC, in a way that the triangles $E_1F_1K_1$ and $E_2F_2K_2$ are similar. Then the six circumcircles of the triangles AE_iF_i, BF_iK_i, and CE_iK_i $(i = 1, 2)$ meet at a common point Z. Moreover, Z is the centre of the spiral similarity that takes the triangle $E_1F_1K_1$ to the triangle $E_2F_2K_2$.

Proof. Firstly, notice that for each $i = 1, 2$, the circumcircles of the triangles AE_iF_i, BF_iK_i, and CK_iE_i have a common point Z_i by MIQUEL's theorem. Moreover, we have

$$\measuredangle(Z_iF_i, Z_iE_i) = \measuredangle(AB, CA), \quad \measuredangle(Z_iK_i, Z_iF_i) = \measuredangle(BC, AB), \quad \measuredangle(Z_iE_i, Z_iK_i) = \measuredangle(CA, BC).$$

This yields that the points Z_1 and Z_2 correspond to each other in similar triangles $E_1F_1K_1$ and $E_2F_2K_2$. Thus, if they coincide, then this common point is indeed the desired centre of a spiral similarity.

Finally, in order to show that $Z_1 = Z_2$, one may notice that $\measuredangle(AB, AZ_1) = \measuredangle(E_1F_1, E_1Z_1) = \measuredangle(E_2F_2, E_2Z_2) = \measuredangle(AB, AZ_2)$ (see Figure 2). Similarly, one has $\measuredangle(BC, BZ_1) = \measuredangle(BC, BZ_2)$ and $\measuredangle(CA, CZ_1) = \measuredangle(CA, CZ_2)$. This yields $Z_1 = Z_2$. □

Now, let P and Q be the feet of the perpendiculars from B and C onto AC and AB, respectively, and let R be the midpoint of BC (see Figure 3). Then R is the circumcentre of the cyclic quadrilateral $BCPQ$. Thus we obtain $\angle APQ = \angle B$ and $\angle RPC = \angle C$, which yields $\angle QPR = \angle A$. Similarly, we show that $\angle PQR = \angle A$. Thus, all interesting triangles are similar to the triangle PQR.

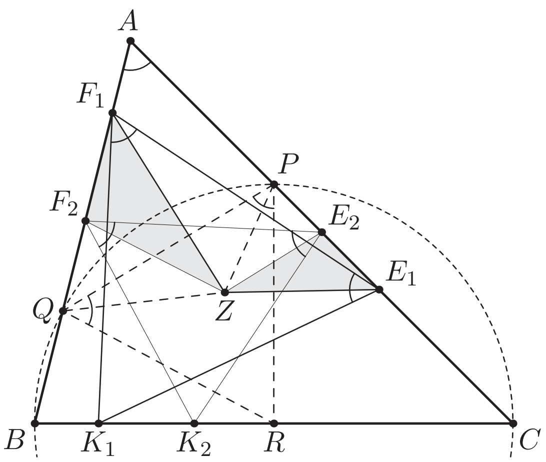


Figure 3

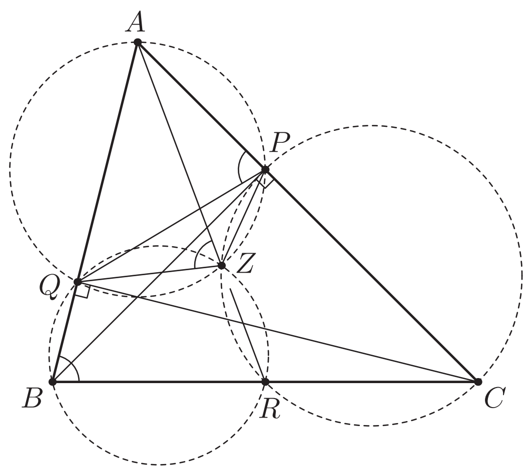


Figure 4

Denote now by Z the common point of the circumcircles of APQ, BQR, and CPR. Let $E_1F_1K_1$ and $E_2F_2K_2$ be two interesting triangles. By the lemma, Z is the centre of any spiral similarity taking one of the triangles $E_1F_1K_1$, $E_2F_2K_2$, and PQR to some other of them. Therefore the triangles ZE_1E_2 and ZF_1F_2 are similar, as well as the triangles ZE_1F_1 and ZPQ. Hence

$$\frac{E_1E_2}{F_1F_2} = \frac{ZE_1}{ZF_1} = \frac{ZP}{ZQ}.$$

Moreover, the equalities $\angle AZQ = \angle APQ = \angle ABC = 180^\circ - \angle QZR$ show that the point Z lies on the line AR (see Figure 4). Therefore the triangles AZP and ACR are similar, as well as the triangles AZQ and ABR. This yields

$$\frac{ZP}{ZQ} = \frac{ZP}{RC} \cdot \frac{RB}{ZQ} = \frac{AZ}{AC} \cdot \frac{AB}{AZ} = \frac{AB}{AC},$$

which completes the solution.

Second method. Now we will start from the fact that ω_1 is tangent to the lines KE and KF (see Figure 5). We prove that if (E, F) is an interesting pair, then

$$\frac{AE}{AB} + \frac{AF}{AC} = 2\cos\angle A. \tag{1}$$

Let Y be the intersection point of the segments BE and CF. The points B, K, and C are collinear, hence applying PASCAL's theorem to the degenerated hexagon $AFFYEE$, we infer that Y lies on the circle ω_1.

Denote by Z the second intersection point of the circumcircle of the triangle BFY with the line BC (see Figure 6). By MIQUEL's theorem, the points C, Z, Y, and E are concyclic. Therefore we obtain

$$BF \cdot AB + CE \cdot AC = BY \cdot BE + CY \cdot CF = BZ \cdot BC + CZ \cdot BC = BC^2\,.$$

On the other hand, $BC^2 = AB^2 + AC^2 - 2AB \cdot AC\cos\angle A$, by the cosine law. Hence

$$(AB - AF) \cdot AB + (AC - AE) \cdot AC = AB^2 + AC^2 - 2AB \cdot AC\cos\angle A\,,$$

which simplifies to the desired equality (1).

Let now (E_1, F_1) and (E_2, F_2) be two interesting pairs of points. Then we get

$$\frac{AE_1}{AB} + \frac{AF_1}{AC} = \frac{AE_2}{AB} + \frac{AF_2}{AC}\,,$$

which gives the desired result.

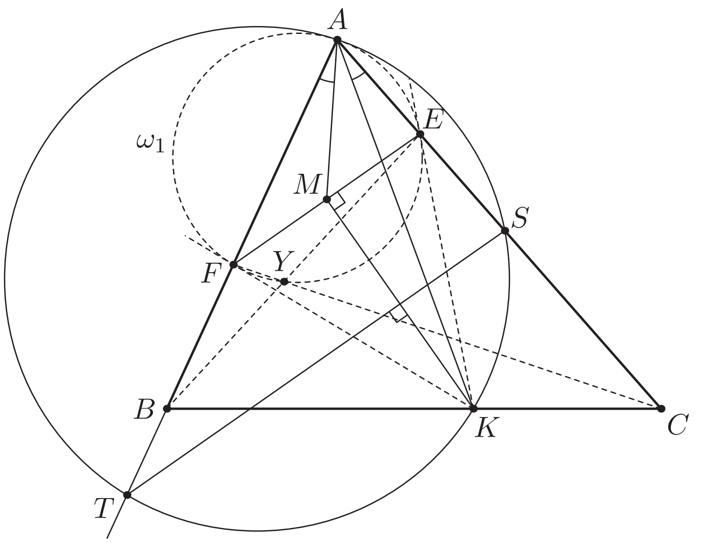


Figure 5

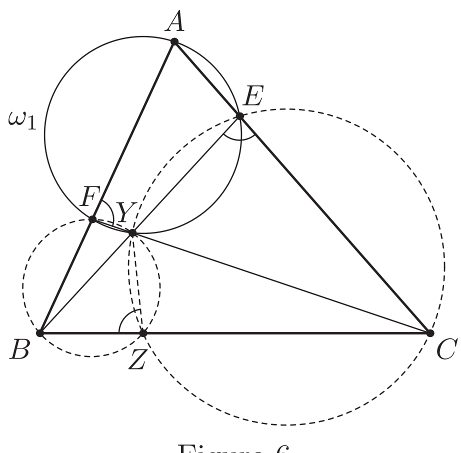


Figure 6

Third method. Again, we make use of the fact that all interesting triangles are similar (and equi-oriented). Let us put the picture onto a complex plane such that A is at the origin, and identify each point with the corresponding complex number.

Let EFK be any interesting triangle. The equalities $\angle KEF = \angle KFE = \angle A$ yield that the ratio $\nu = \frac{K-E}{F-E}$ is the same for all interesting triangles. This in turn means that the numbers E, F, and K satisfy the linear equation

$$K = \mu E + \nu F, \qquad \text{where} \quad \mu = 1 - \nu. \tag{2}$$

Now let us choose the points X and Y on the rays AB and AC, respectively, so that $\angle CXA = \angle AYB = \angle A = \angle KEF$ (see Figure 7). Then each of the triangles AXC and YAB is similar to any interesting triangle, which also means that

$$C = \mu A + \nu X = \nu X \quad \text{and} \quad B = \mu Y + \nu A = \mu Y. \tag{3}$$

Moreover, one has $X/Y = \overline{C/B}$.

Since the points E, F, and K lie on AC, AB, and BC, respectively, one gets

$$E = \rho Y, \quad F = \sigma X, \quad \text{and} \quad K = \lambda B + (1-\lambda)C$$

for some real ρ, σ, and λ. In view of (3), the equation (2) now reads $\lambda B + (1-\lambda)C = K = \mu E + \nu F = \rho B + \sigma C$, or

$$(\lambda - \rho)B = (\sigma + \lambda - 1)C.$$

Since the nonzero complex numbers B and C have different arguments, the coefficients in the brackets vanish, so $\rho = \lambda$ and $\sigma = 1 - \lambda$. Therefore,

$$\frac{E}{Y} + \frac{F}{X} = \rho + \sigma = 1. \tag{4}$$

Now, if (E_1, F_1) and (E_2, F_2) are two distinct interesting pairs, one may apply (4) to both pairs. Subtracting, we get

$$\frac{E_1 - E_2}{Y} = \frac{F_2 - F_1}{X}, \qquad \text{so} \quad \frac{E_1 - E_2}{F_2 - F_1} = \frac{Y}{X} = \frac{\overline{B}}{\overline{C}}.$$

Taking absolute values provides the required result.

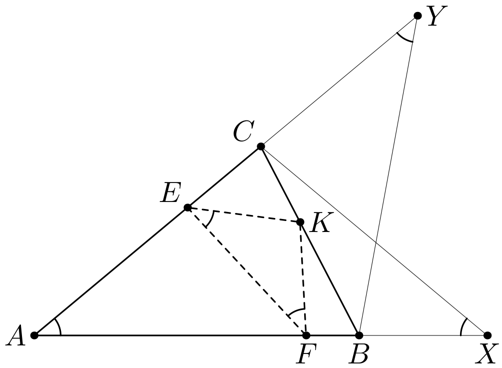


Figure 7

Comment 1. One may notice that the triangle PQR is also interesting.

Comment 2. In order to prove that $\angle KEF = \angle KFE = \angle A$, one may also use the following well-known fact:

Let AEF be a triangle with $AE \neq AF$, and let K be the common point of the symmedian taken from A and the perpendicular bisector of EF. Then the lines KE and KF are tangent to the circumcircle ω_1 of the triangle AEF.

In this case, however, one needs to deal with the case $AE = AF$ separately.

Solution 2. Let (E, F) be an interesting pair. This time we prove that

$$\frac{AM}{AK} = \cos\angle A. \tag{5}$$

As in Solution 1, we introduce the circle ω passing through the points K, S, A, and T, together with the points N and L at which the line AM intersect the line ST and the circle ω for the second time, respectively. Let moreover O be the centre of ω (see Figures 8 and 9). As in Solution 1, we note that N is the midpoint of ST and show that $KL \parallel ST$, which implies $\angle FAM = \angle EAK$.

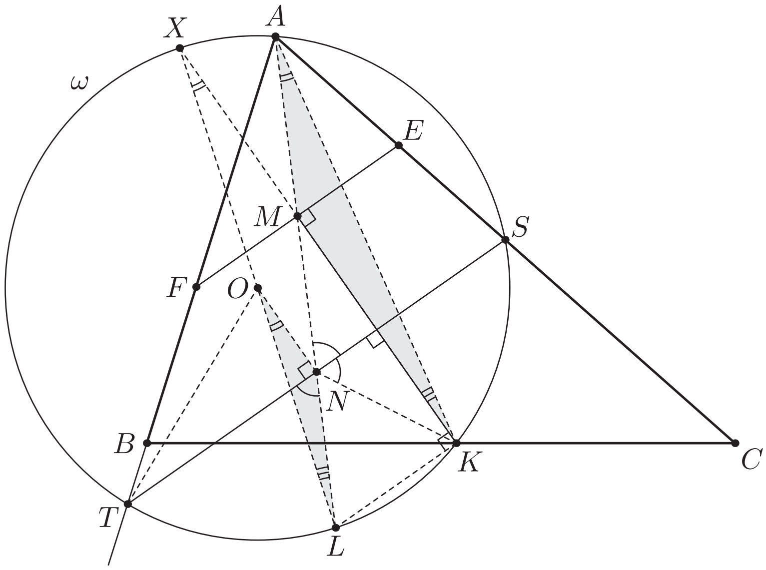


Figure 8

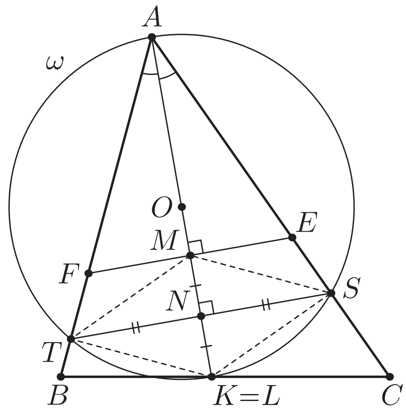


Figure 9

Suppose now that $K \neq L$ (see Figure 8). Then $KL \parallel ST$, and consequently the lines KM and KL are perpendicular. It implies that the lines LO and KM meet at a point X lying on the circle ω. Since the lines ON and XM are both perpendicular to the line ST, they are parallel to each other, and hence $\angle LON = \angle LXK = \angle MAK$. On the other hand, $\angle OLN = \angle MKA$, so we infer that triangles NOL and MAK are similar. This yields

$$\frac{AM}{AK} = \frac{ON}{OL} = \frac{ON}{OT} = \cos\angle TON = \cos\angle A.$$

If, on the other hand, $K = L$, then the points A, M, N, and K lie on a common line, and this line is the perpendicular bisector of ST (see Figure 9). This implies that AK is a diameter of ω, which yields $AM = 2OK - 2NK = 2ON$. So also in this case we obtain

$$\frac{AM}{AK} = \frac{2ON}{2OT} = \cos\angle TON = \cos\angle A.$$

Thus (5) is proved.

Let P and Q be the feet of the perpendiculars from B and C onto AC and AB, respectively (see Figure 10). We claim that the point M lies on the line PQ. Consider now the composition of the dilatation with factor $\cos\angle A$ and centre A, and the reflection with respect to the angle bisector of $\angle BAC$. This transformation is a similarity that takes B, C, and K to P, Q, and M, respectively. Since K lies on the line BC, the point M lies on the line PQ.

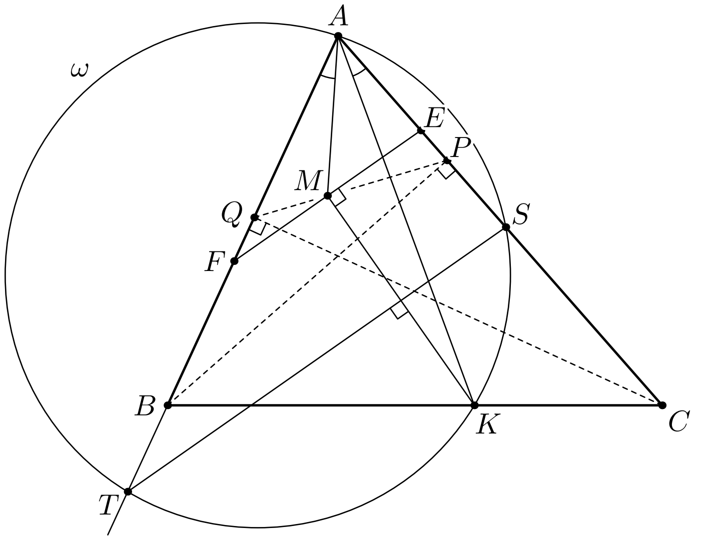


Figure 10

Suppose that $E \neq P$. Then also $F \neq Q$, and by MENELAUS' theorem, we obtain

$$\frac{AQ}{FQ} \cdot \frac{FM}{EM} \cdot \frac{EP}{AP} = 1\,.$$

Using the similarity of the triangles APQ and ABC, we infer that

$$\frac{EP}{FQ} = \frac{AP}{AQ} = \frac{AB}{AC}\,, \quad \text{and hence} \quad \frac{EP}{AB} = \frac{FQ}{AC}\,.$$

The last equality holds obviously also in case $E = P$, because then $F = Q$. Moreover, since the line PQ intersects the *segment* EF, we infer that the point E lies on the segment AP if and only if the point F lies outside of the segment AQ.

Let now (E_1, F_1) and (E_2, F_2) be two interesting pairs. Then we obtain

$$\frac{E_1P}{AB} = \frac{F_1Q}{AC} \qquad \text{and} \qquad \frac{E_2P}{AB} = \frac{F_2Q}{AC}\,.$$

If P lies between the points E_1 and E_2, we add the equalities above, otherwise we subtract them. In any case we obtain

$$\frac{E_1E_2}{AB} = \frac{F_1F_2}{AC}\,,$$

which completes the solution.

G7. Let ABC be a triangle with circumcircle Ω and incentre I. Let the line passing through I and perpendicular to CI intersect the segment BC and the arc BC (not containing A) of Ω at points U and V, respectively. Let the line passing through U and parallel to AI intersect AV at X, and let the line passing through V and parallel to AI intersect AB at Y. Let W and Z be the midpoints of AX and BC, respectively. Prove that if the points I, X, and Y are collinear, then the points I, W, and Z are also collinear.

(U.S.A.)

Solution 1. We start with some general observations. Set $\alpha = \angle A/2$, $\beta = \angle B/2$, $\gamma = \angle C/2$. Then obviously $\alpha + \beta + \gamma = 90^\circ$. Since $\angle UIC = 90^\circ$, we obtain $\angle IUC = \alpha + \beta$. Therefore $\angle BIV = \angle IUC - \angle IBC = \alpha = \angle BAI = \angle BYV$, which implies that the points B, Y, I, and V lie on a common circle (see Figure 1).

Assume now that the points I, X and Y are collinear. We prove that $\angle YIA = 90^\circ$.

Let the line XU intersect AB at N. Since the lines AI, UX, and VY are parallel, we get

$$\frac{NX}{AI} = \frac{YN}{YA} = \frac{VU}{VI} = \frac{XU}{AI},$$

implying $NX = XU$. Moreover, $\angle BIU = \alpha = \angle BNU$. This implies that the quadrilateral $BUIN$ is cyclic, and since BI is the angle bisector of $\angle UBN$, we infer that $NI = UI$. Thus in the isosceles triangle NIU, the point X is the midpoint of the base NU. This gives $\angle IXN = 90^\circ$, i.e., $\angle YIA = 90^\circ$.

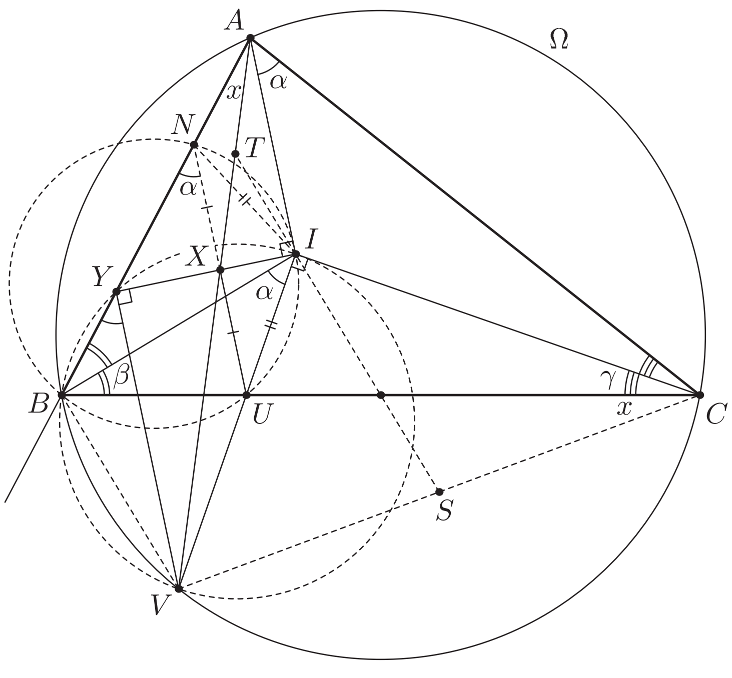


Figure 1

Let S be the midpoint of the segment VC. Let moreover T be the intersection point of the lines AX and SI, and set $x = \angle BAV = \angle BCV$. Since $\angle CIA = 90^\circ + \beta$ and $SI = SC$, we obtain

$$\angle TIA = 180^\circ - \angle AIS = 90^\circ - \beta - \angle CIS = 90^\circ - \beta - \gamma - x = \alpha - x = \angle TAI,$$

which implies that $TI = TA$. Therefore, since $\angle XIA = 90^\circ$, the point T is the midpoint of AX, i.e., $T = W$.

To complete our solution, it remains to show that the intersection point of the lines IS and BC coincide with the midpoint of the segment BC. But since S is the midpoint of the segment VC, it suffices to show that the lines BV and IS are parallel.

Since the quadrilateral $BYIV$ is cyclic, $\angle VBI = \angle VYI = \angle YIA = 90^\circ$. This implies that BV is the external angle bisector of the angle ABC, which yields $\angle VAC = \angle VCA$. Therefore $2\alpha - x = 2\gamma + x$, which gives $\alpha = \gamma + x$. Hence $\angle SCI = \alpha$, so $\angle VSI = 2\alpha$.

On the other hand, $\angle BVC = 180^\circ - \angle BAC = 180^\circ - 2\alpha$, which implies that the lines BV and IS are parallel. This completes the solution.

Solution 2. As in Solution 1, we first prove that the points B, Y, I, V lie on a common circle and $\angle YIA = 90^\circ$. The remaining part of the solution is based on the following lemma, which holds true for any triangle ABC, not necessarily with the property that I, X, Y are collinear.

Lemma. Let ABC be the triangle inscribed in a circle Γ and let I be its incentre. Assume that the line passing through I and perpendicular to the line AI intersects the side AB at the point Y. Let the circumcircle of the triangle BYI intersect the circle Γ for the second time at V, and let the excircle of the triangle ABC opposite to the vertex A be tangent to the side BC at E. Then

$$\angle BAV = \angle CAE\,.$$

Proof. Let ρ be the composition of the inversion with centre A and radius $\sqrt{AB \cdot AC}$, and the symmetry with respect to AI. Clearly, ρ interchanges B and C.

Let J be the excentre of the triangle ABC opposite to A (see Figure 2). Then we have $\angle JAC = \angle BAI$ and $\angle JCA = 90^\circ + \gamma = \angle BIA$, so the triangles ACJ and AIB are similar, and therefore $AB \cdot AC = AI \cdot AJ$. This means that ρ interchanges I and J. Moreover, since Y lies on AB and $\angle AIY = 90^\circ$, the point $Y' = \rho(Y)$ lies on AC, and $\angle JY'A = 90^\circ$. Thus ρ maps the circumcircle γ of the triangle BYI to a circle γ' with diameter JC.

Finally, since V lies on both Γ and γ, the point $V' = \rho(V)$ lies on the line $\rho(\Gamma) = AB$ as well as on γ', which in turn means that $V' = E$. This implies the desired result. □

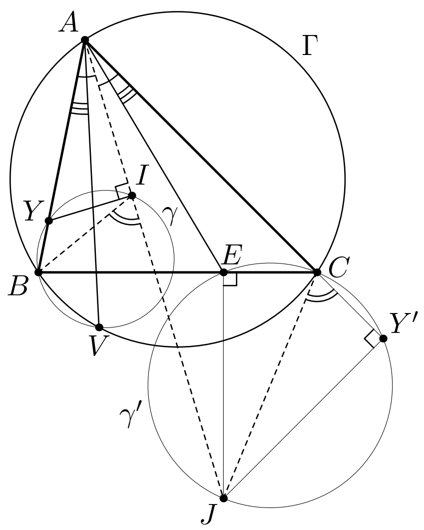


Figure 2

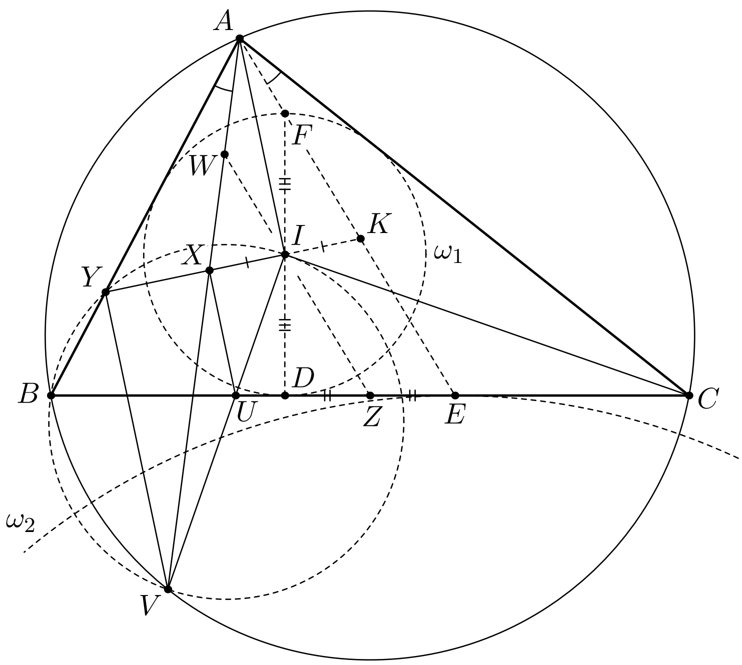


Figure 3

Now we turn to the solution of the problem.

Assume that the incircle ω_1 of the triangle ABC is tangent to BC at D, and let the excircle ω_2 of the triangle ABC opposite to the vertex A touch the side BC at E (see Figure 3). The homothety with centre A that takes ω_2 to ω_1 takes the point E to some point F, and the tangent to ω_1 at F is parallel to BC. Therefore DF is a diameter of ω_1. Moreover, Z is the midpoint of DE. This implies that the lines IZ and FE are parallel.

Let $K = YI \cap AE$. Since $\angle YIA = 90^\circ$, the lemma yields that I is the midpoint of XK. This implies that the segments IW and AK are parallel. Therefore, the points W, I and Z are collinear.

Comment 1. The properties $\angle YIA = 90^\circ$ and $VA = VC$ can be established in various ways. The main difficulty of the problem seems to find out how to use these properties in connection to the points W and Z.

In Solution 2 this principal part is more or less covered by the lemma, for which we have presented a direct proof. On the other hand, this lemma appears to be a combination of two well-known facts; let us formulate them in terms of the lemma statement.

Let the line IY intersect AC at P (see Figure 4). The first fact states that the circumcircle ω of the triangle VYP is tangent to the segments AB and AC, as well as to the circle Γ. The second fact states that for such a circle, the angles BAV and CAE are equal.

The awareness of this lemma may help a lot in solving this problem; so the Jury might also consider a variation of the proposed problem, for which the lemma does not seem to be useful; see Comment 3.

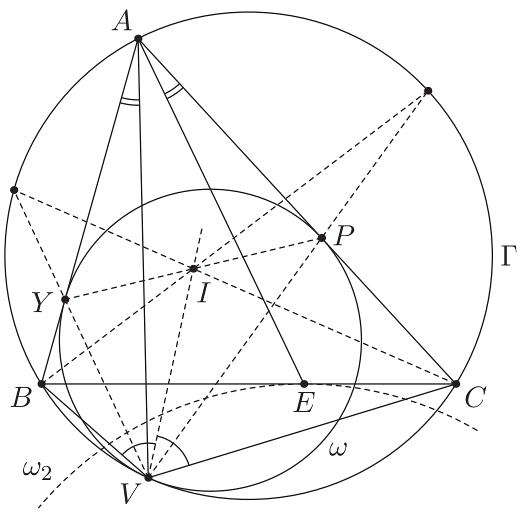


Figure 4

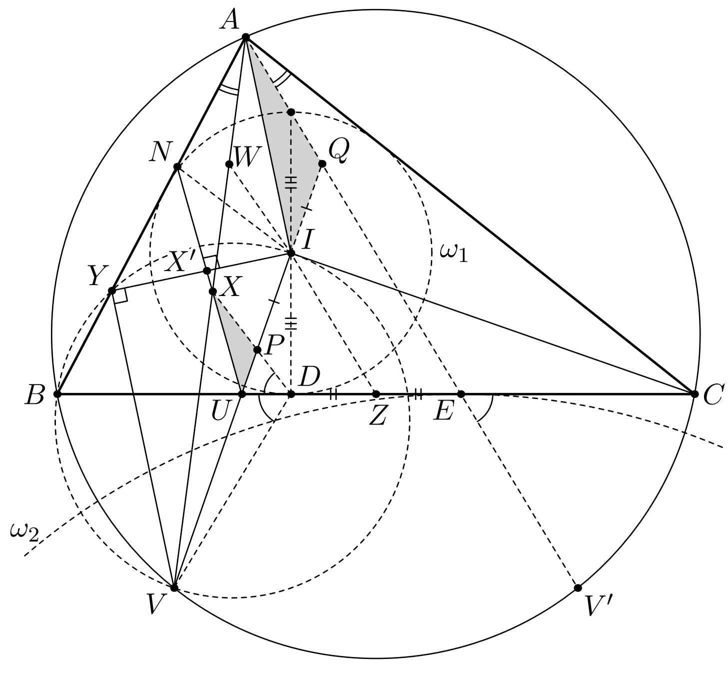


Figure 5

Comment 2. The proposed problem stated the equivalence: the point I lies on the line XY if and only if I lies on the line WZ. Here we sketch the proof of the "if" part (see Figure 5).

As in Solution 2, let BC touch the circles ω_1 and ω_2 at D and E, respectively. Since $IZ \parallel AE$ and W lies on IZ, the line DX is also parallel to AE. Therefore, the triangles XUP and AIQ are similar. Moreover, the line DX is symmetric to AE with respect to I, so $IP = IQ$, where $P = UV \cap XD$ and $Q = UV \cap AE$. Thus we obtain

$$\frac{UV}{VI} = \frac{UX}{IA} = \frac{UP}{IQ} = \frac{UP}{IP}.$$

So the pairs IU and PV are harmonic conjugates, and since $\angle UDI = 90^\circ$, we get $\angle VDB = \angle BDX = \angle BEA$. Therefore the point V' symmetric to V with respect to the perpendicular bisector of BC lies on the line AE. So we obtain $\angle BAV = \angle CAE$.

The rest can be obtained by simply reversing the arguments in Solution 2. The points B, V, I, and Y are concyclic. The lemma implies that $\angle YIA = 90^\circ$. Moreover, the points B, U, I, and N, where $N = UX \cap AB$, lie on a common circle, so $IN = IU$. Since $IY \perp UN$, the point $X' = IY \cap UN$ is the midpoint of UN. But in the trapezoid $AYVI$, the line XU is parallel to the sides AI and YV, so $NX = UX'$. This yields $X = X'$.

The reasoning presented in Solution 1 can also be reversed, but it requires a lot of technicalities. Therefore the Problem Selection Committee proposes to consider only the "only if" part of the original proposal, which is still challenging enough.

Comment 3. The Jury might also consider the following variation of the proposed problem.

Let ABC be a triangle with circumcircle Ω and incentre I. Let the line through I perpendicular to CI intersect the segment BC and the arc BC (not containing A) of Ω at U and V, respectively. Let the line through U parallel to AI intersect AV at X. Prove that if the lines XI and AI are perpendicular, then the midpoint of the segment AC lies on the line XI (see Figure 6).

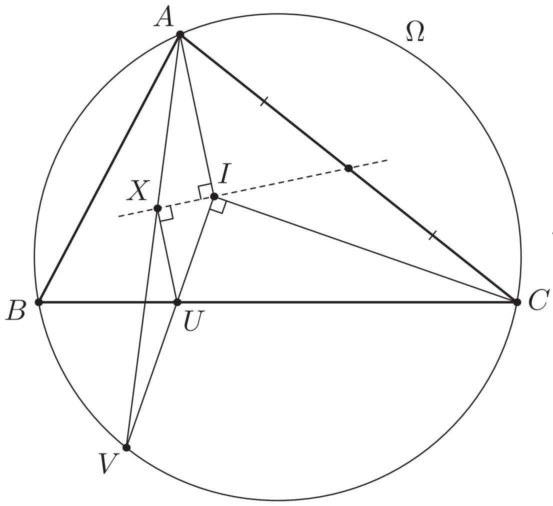


Figure 6

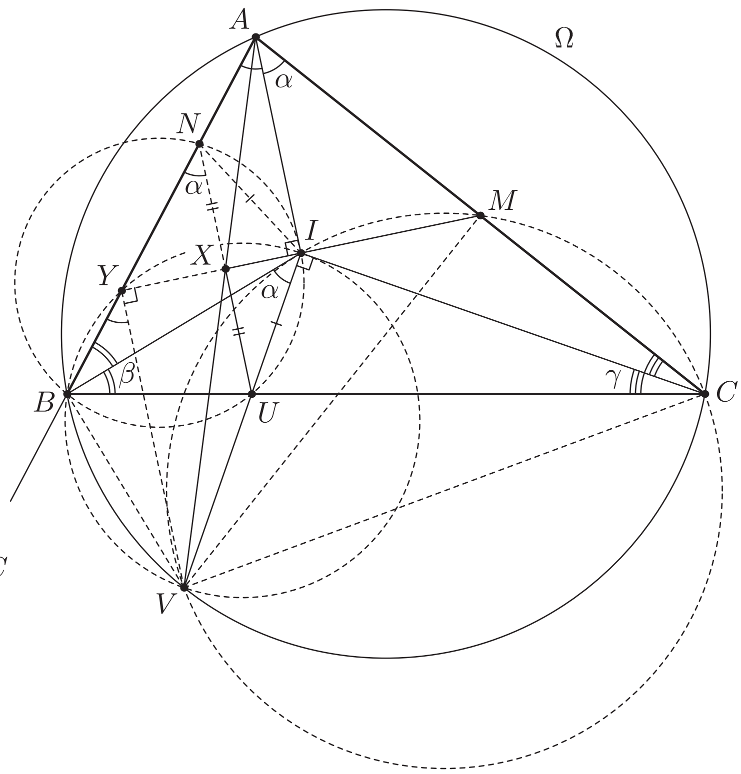


Figure 7

Since the solution contains the arguments used above, we only sketch it.

Let $N = XU \cap AB$ (see Figure 7). Then $\angle BNU = \angle BAI = \angle BIU$, so the points B, U, I, and N lie on a common circle. Therefore $IU = IN$, and since $IX \perp NU$, it follows that $NX = XU$.

Now set $Y = XI \cap AB$. The equality $NX = XU$ implies that

$$\frac{VX}{VA} = \frac{XU}{AI} = \frac{NX}{AI} = \frac{YX}{YI},$$

and therefore $YV \parallel AI$. Hence $\angle BYV = \angle BAI = \angle BIV$, so the points B, V, I, Y are concyclic. Next we have $IY \perp YV$, so $\angle IBV = 90^\circ$. This implies that BV is the external angle bisector of the angle ABC, which gives $\angle VAC = \angle VCA$.

So in order to show that $M = XI \cap AC$ is the midpoint of AC, it suffices to prove that $\angle VMC = 90^\circ$. But this follows immediately from the observation that the points V, C, M, and I are concyclic, as $\angle MIV = \angle YBV = 180^\circ - \angle ACV$.

The converse statement is also true, but its proof requires some technicalities as well.

Number Theory

N1. Let $n \geqslant 2$ be an integer, and let A_n be the set

$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z},\ 0 \leqslant k < n\}.$$

Determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of A_n.

(Serbia)

Answer. $(n-2)2^n + 1$.

Solution 1.

Part I. First we show that every integer greater than $(n-2)2^n + 1$ can be represented as such a sum. This is achieved by induction on n.

For $n = 2$, the set A_n consists of the two elements 2 and 3. Every positive integer m except for 1 can be represented as the sum of elements of A_n in this case: as $m = 2 + 2 + \cdots + 2$ if m is even, and as $m = 3 + 2 + 2 + \cdots + 2$ if m is odd.

Now consider some $n > 2$, and take an integer $m > (n-2)2^n + 1$. If m is even, then consider

$$\frac{m}{2} \geqslant \frac{(n-2)2^n + 2}{2} = (n-2)2^{n-1} + 1 > (n-3)2^{n-1} + 1.$$

By the induction hypothesis, there is a representation of the form

$$\frac{m}{2} = (2^{n-1} - 2^{k_1}) + (2^{n-1} - 2^{k_2}) + \cdots + (2^{n-1} - 2^{k_r})$$

for some k_i with $0 \leqslant k_i < n-1$. It follows that

$$m = (2^n - 2^{k_1+1}) + (2^n - 2^{k_2+1}) + \cdots + (2^n - 2^{k_r+1}),$$

giving us the desired representation as a sum of elements of A_n. If m is odd, we consider

$$\frac{m - (2^n - 1)}{2} > \frac{(n-2)2^n + 1 - (2^n - 1)}{2} = (n-3)2^{n-1} + 1.$$

By the induction hypothesis, there is a representation of the form

$$\frac{m - (2^n - 1)}{2} = (2^{n-1} - 2^{k_1}) + (2^{n-1} - 2^{k_2}) + \cdots + (2^{n-1} - 2^{k_r})$$

for some k_i with $0 \leqslant k_i < n-1$. It follows that

$$m = (2^n - 2^{k_1+1}) + (2^n - 2^{k_2+1}) + \cdots + (2^n - 2^{k_r+1}) + (2^n - 1),$$

giving us the desired representation of m once again.

Part II. It remains to show that there is no representation for $(n-2)2^n + 1$. Let N be the smallest positive integer that satisfies $N \equiv 1 \pmod{2^n}$, and which can be represented as a sum of elements of A_n. Consider a representation of N, i.e.,

$$N = (2^n - 2^{k_1}) + (2^n - 2^{k_2}) + \cdots + (2^n - 2^{k_r}), \tag{1}$$

where $0 \leqslant k_1, k_2, \ldots, k_r < n$. Suppose first that two of the terms in the sum are the same, i.e., $k_i = k_j$ for some $i \neq j$. If $k_i = k_j = n-1$, then we can simply remove these two terms to get a representation for

$$N - 2(2^n - 2^{n-1}) = N - 2^n$$

as a sum of elements of A_n, which contradicts our choice of N. If $k_i = k_j = k < n-1$, replace the two terms by $2^n - 2^{k+1}$, which is also an element of A_n, to get a representation for

$$N - 2(2^n - 2^k) + 2^n - 2^{k+1} = N - 2^n.$$

This is a contradiction once again. Therefore, all k_i have to be distinct, which means that

$$2^{k_1} + 2^{k_2} + \cdots + 2^{k_r} \leqslant 2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} = 2^n - 1.$$

On the other hand, taking (1) modulo 2^n, we find

$$2^{k_1} + 2^{k_2} + \cdots + 2^{k_r} \equiv -N \equiv -1 \pmod{2^n}.$$

Thus we must have $2^{k_1} + 2^{k_2} + \cdots + 2^{k_r} = 2^n - 1$, which is only possible if each element of $\{0, 1, \ldots, n-1\}$ occurs as one of the k_i. This gives us

$$N = n2^n - (2^0 + 2^1 + \cdots + 2^{n-1}) = (n-1)2^n + 1.$$

In particular, this means that $(n-2)2^n + 1$ cannot be represented as a sum of elements of A_n.

Solution 2. The fact that $m = (n-2)2^n + 1$ cannot be represented as a sum of elements of A_n can also be shown in other ways. We prove the following statement by induction on n:

Claim. If a, b are integers with $a \geqslant 0$, $b \geqslant 1$, and $a + b < n$, then $a2^n + b$ cannot be written as a sum of elements of A_n.

Proof. The claim is clearly true for $n = 2$ (since $a = 0$, $b = 1$ is the only possibility). For $n > 2$, assume that there exist integers a, b with $a \geqslant 0$, $b \geqslant 1$ and $a + b < n$ as well as elements $m_1, m_2, \ldots, m_r$ of A_n such that

$$a2^n + b = m_1 + m_2 + \cdots + m_r.$$

We can suppose, without loss of generality, that $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_r$. Let ℓ be the largest index for which $m_\ell = 2^n - 1$ ($\ell = 0$ if $m_1 \neq 2^n - 1$). Clearly, ℓ and b must have the same parity. Now

$$(a - \ell)2^n + (b + \ell) = m_{\ell+1} + m_{\ell+2} + \cdots + m_r$$

and thus

$$(a - \ell)2^{n-1} + \frac{b + \ell}{2} = \frac{m_{\ell+1}}{2} + \frac{m_{\ell+2}}{2} + \cdots + \frac{m_r}{2}.$$

Note that $m_{\ell+1}/2, m_{\ell+2}/2, \ldots, m_r/2$ are elements of A_{n-1}. Moreover, $a - \ell$ and $(b + \ell)/2$ are integers, and $(b + \ell)/2 \geqslant 1$. If $a - \ell$ was negative, then we would have

$$a2^n + b \geqslant \ell(2^n - 1) \geqslant (a + 1)(2^n - 1) = a2^n + 2^n - a - 1,$$

thus $n \geqslant a + b + 1 \geqslant 2^n$, which is impossible. So $a - \ell \geqslant 0$. By the induction hypothesis, we must have $a - \ell + \frac{b+\ell}{2} \geqslant n - 1$, which gives us a contradiction, since

$$a - \ell + \frac{b + \ell}{2} \leqslant a - \ell + b + \ell - 1 = a + b - 1 < n - 1. \qquad \square$$

Considering the special case $a = n - 2$, $b = 1$ now completes the proof.

Solution 3. Denote by B_n the set of all positive integers that can be written as a sum of elements of A_n. In this solution, we explicitly describe all the numbers in B_n by an argument similar to the first solution.

For a positive integer n, we denote by $\sigma_2(n)$ the sum of its digits in the binary representation. Notice that every positive integer m has a unique representation of the form $m = s2^n - t$ with some positive integer s and $0 \leqslant t \leqslant 2^n - 1$.

Lemma. For any two integers $s \geqslant 1$ and $0 \leqslant t \leqslant 2^n - 1$, the number $m = s2^n - t$ belongs to B_n if and only if $s \geqslant \sigma_2(t)$.

Proof. For $t = 0$, the statement of the Lemma is obvious, since $m = 2s \cdot (2^n - 2^{n-1})$.

Now suppose that $t \geqslant 1$, and let

$$t = 2^{k_1} + \cdots + 2^{k_\sigma} \qquad (0 \leqslant k_1 < \cdots < k_\sigma \leqslant n-1, \quad \sigma = \sigma_2(t))$$

be its binary expansion. If $s \geqslant \sigma$, then $m \in B_n$ since

$$m = (s-\sigma)2^n + (\sigma 2^n - t) = 2(s-\sigma) \cdot (2^n - 2^{n-1}) + \sum_{i=1}^{\sigma}(2^n - 2^{k_i}).$$

Assume now that there exist integers s and t with $1 \leqslant s < \sigma_2(t)$ and $0 \leqslant t \leqslant 2^n - 1$ such that the number $m = s2^n - t$ belongs to B_n. Among all such instances, choose the one for which m is smallest, and let

$$m = \sum_{i=1}^{d}(2^n - 2^{\ell_i}) \qquad (0 \leqslant \ell_i \leqslant n-1)$$

be the corresponding representation. If all the ℓ_i's are distinct, then $\sum_{i=1}^{d} 2^{\ell_i} \leqslant \sum_{j=0}^{n-1} 2^j = 2^n - 1$, so one has $s = d$ and $t = \sum_{i=1}^{d} 2^{\ell_i}$, whence $s = d = \sigma_2(t)$; this is impossible. Therefore, two of the ℓ_i's must be equal, say $\ell_{d-1} = \ell_d$. Then $m \geqslant 2(2^n - 2^{\ell_d}) \geqslant 2^n$, so $s \geqslant 2$.

Now we claim that the number $m' = m - 2^n = (s-1)2^n - t$ also belongs to B_n, which contradicts the minimality assumption. Indeed, one has

$$(2^n - 2^{\ell_{d-1}}) + (2^n - 2^{\ell_d}) = 2(2^n - 2^{\ell_d}) = 2^n + (2^n - 2^{\ell_d+1}),$$

so

$$m' = \sum_{i=1}^{d-2}(2^n - 2^{\ell_i}) + (2^n - 2^{\ell_d+1})$$

is the desired representation of m' (if $\ell_d = n-1$, then the last summand is simply omitted). This contradiction finishes the proof. $\square$

By our lemma, the largest number M which does not belong to B_n must have the form

$$m_t = (\sigma_2(t) - 1)2^n - t$$

for some t with $1 \leqslant t \leqslant 2^n - 1$, so M is just the largest of these numbers. For $t_0 = 2^n - 1$ we have $m_{t_0} = (n-1)2^n - (2^n - 1) = (n-2)2^n + 1$; for every other value of t one has $\sigma_2(t) \leqslant n-1$, thus $m_t \leqslant (\sigma(t) - 1)2^n \leqslant (n-2)2^n < m_{t_0}$. This means that $M = m_{t_0} = (n-2)2^n + 1$.

N2. Determine all pairs (x, y) of positive integers such that

$$\sqrt[3]{7x^2 - 13xy + 7y^2} = |x - y| + 1 . \tag{1}$$

(U.S.A.)

Answer. Either $(x, y) = (1, 1)$ or $\{x, y\} = \{m^3 + m^2 - 2m - 1, m^3 + 2m^2 - m - 1\}$ for some positive integer $m \geqslant 2$.

Solution. Let (x, y) be any pair of positive integers solving (1). We shall prove that it appears in the list displayed above. The converse assertion that all these pairs do actually satisfy (1) either may be checked directly by means of a somewhat laborious calculation, or it can be seen by going in reverse order through the displayed equations that follow.

In case $x = y$ the given equation reduces to $x^{2/3} = 1$, which is equivalent to $x = 1$, whereby he have found the first solution.

To find the solutions with $x \neq y$ we may assume $x > y$ due to symmetry. Then the integer $n = x - y$ is positive and (1) may be rewritten as

$$\sqrt[3]{7(y + n)^2 - 13(y + n)y + 7y^2} = n + 1 .$$

Raising this to the third power and simplifying the result one obtains

$$y^2 + yn = n^3 - 4n^2 + 3n + 1 .$$

To complete the square on the left hand side, we multiply by 4 and add n^2, thus getting

$$(2y + n)^2 = 4n^3 - 15n^2 + 12n + 4 = (n - 2)^2(4n + 1) .$$

This shows that the cases $n = 1$ and $n = 2$ are impossible, whence $n > 2$, and $4n + 1$ is the square of the rational number $\frac{2y+n}{n-2}$. Consequently, it has to be a perfect square, and, since it is odd as well, there has to exist some nonnegative integer m such that $4n + 1 = (2m + 1)^2$, i.e.

$$n = m^2 + m .$$

Notice that $n > 2$ entails $m \geqslant 2$. Substituting the value of n just found into the previous displayed equation we arrive at

$$(2y + m^2 + m)^2 = (m^2 + m - 2)^2(2m + 1)^2 = (2m^3 + 3m^2 - 3m - 2)^2 .$$

Extracting square roots and taking $2m^3 + 3m^2 - 3m - 2 = (m - 1)(2m^2 + 5m + 2) > 0$ into account we derive $2y + m^2 + m = 2m^3 + 3m^2 - 3m - 2$, which in turn yields

$$y = m^3 + m^2 - 2m - 1 .$$

Notice that $m \geqslant 2$ implies that $y = (m^3 - 1) + (m - 2)m$ is indeed positive, as it should be. In view of $x = y + n = y + m^2 + m$ it also follows that

$$x = m^3 + 2m^2 - m - 1 ,$$

and that this integer is positive as well.

Comment. Alternatively one could ask to find all pairs (x, y) of – not necessarily positive – integers solving (1). The answer to that question is a bit nicer than the answer above: the set of solutions are now described by

$$\{x, y\} = \{m^3 + m^2 - 2m - 1, m^3 + 2m^2 - m - 1\} ,$$

where m varies through $\mathbb{Z}$. This may be shown using essentially the same arguments as above. We finally observe that the pair $(x, y) = (1, 1)$, that appears to be sporadic above, corresponds to $m = -1$.

N3. A coin is called a *Cape Town coin* if its value is $1/n$ for some positive integer n. Given a collection of Cape Town coins of total value at most $99 + \frac{1}{2}$, prove that it is possible to split this collection into at most 100 groups each of total value at most 1.

(Luxembourg)

Solution. We will show that for every positive integer N any collection of Cape Town coins of total value at most $N - \frac{1}{2}$ can be split into N groups each of total value at most 1. The problem statement is a particular case for $N = 100$.

We start with some preparations. If several given coins together have a total value also of the form $\frac{1}{k}$ for a positive integer k, then we may merge them into one new coin. Clearly, if the resulting collection can be split in the required way then the initial collection can also be split.

After each such merging, the total number of coins decreases, thus at some moment we come to a situation when no more merging is possible. At this moment, for every even k there is at most one coin of value $\frac{1}{k}$ (otherwise two such coins may be merged), and for every odd $k > 1$ there are at most $k - 1$ coins of value $\frac{1}{k}$ (otherwise k such coins may also be merged).

Now, clearly, each coin of value 1 should form a single group; if there are d such coins then we may remove them from the collection and replace N by $N - d$. So from now on we may assume that there are no coins of value 1.

Finally, we may split all the coins in the following way. For each $k = 1, 2, \ldots, N$ we put all the coins of values $\frac{1}{2k-1}$ and $\frac{1}{2k}$ into a group G_k; the total value of G_k does not exceed

$$(2k-2) \cdot \frac{1}{2k-1} + \frac{1}{2k} < 1.$$

It remains to distribute the "small" coins of values which are less than $\frac{1}{2N}$; we will add them one by one. In each step, take any remaining small coin. The total value of coins in the groups at this moment is at most $N - \frac{1}{2}$, so there exists a group of total value at most $\frac{1}{N}\left(N - \frac{1}{2}\right) = 1 - \frac{1}{2N}$; thus it is possible to put our small coin into this group. Acting so, we will finally distribute all the coins.

Comment 1. The algorithm may be modified, at least the step where one distributes the coins of values $\geqslant \frac{1}{2N}$. One different way is to put into G_k all the coins of values $\frac{1}{(2k-1)2^s}$ for all integer $s \geqslant 0$. One may easily see that their total value also does not exceed 1.

Comment 2. The original proposal also contained another part, suggesting to show that a required splitting may be impossible if the total value of coins is at most 100. There are many examples of such a collection, e.g. one may take 98 coins of value 1, one coin of value $\frac{1}{2}$, two coins of value $\frac{1}{3}$, and four coins of value $\frac{1}{5}$.

The Problem Selection Committee thinks that this part is less suitable for the competition.

N4. Let $n > 1$ be a given integer. Prove that infinitely many terms of the sequence $(a_k)_{k\geqslant 1}$, defined by

$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor,$$

are odd. (For a real number x, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.)

(Hong Kong)

Solution 1. If n is odd, let $k = n^m$ for $m = 1, 2, \ldots$. Then $a_k = n^{n^m - m}$, which is odd for each m.

Henceforth, assume that n is even, say $n = 2t$ for some integer $t \geqslant 1$. Then, for any $m \geqslant 2$, the integer $n^{2^m} - 2^m = 2^m(2^{2^m - m} \cdot t^{2^m} - 1)$ has an odd prime divisor p, since $2^m - m > 1$. Then, for $k = p \cdot 2^m$, we have

$$n^k = (n^{2^m})^p \equiv (2^m)^p = (2^p)^m \equiv 2^m,$$

where the congruences are taken modulo p (recall that $2^p \equiv 2 \pmod p$, by FERMAT's little theorem). Also, from $n^k - 2^m < n^k < n^k + 2^m(p-1)$, we see that the fraction $\dfrac{n^k}{k}$ lies strictly between the consecutive integers $\dfrac{n^k - 2^m}{p \cdot 2^m}$ and $\dfrac{n^k + 2^m(p-1)}{p \cdot 2^m}$, which gives

$$\left\lfloor \frac{n^k}{k} \right\rfloor = \frac{n^k - 2^m}{p \cdot 2^m}.$$

We finally observe that $\dfrac{n^k - 2^m}{p \cdot 2^m} = \dfrac{\frac{n^k}{2^m} - 1}{p}$ is an odd integer, since the integer $\dfrac{n^k}{2^m} - 1$ is odd (recall that $k > m$). Note that for different values of m, we get different values of k, due to the different powers of 2 in the prime factorisation of k.

Solution 2. Treat the (trivial) case when n is odd as in Solution 1.

Now assume that n is even and $n > 2$. Let p be a prime divisor of $n - 1$.

Proceed by induction on i to prove that p^{i+1} is a divisor of $n^{p^i} - 1$ for every $i \geqslant 0$. The case $i = 0$ is true by the way in which p is chosen. Suppose the result is true for some $i \geqslant 0$. The factorisation

$$n^{p^{i+1}} - 1 = (n^{p^i} - 1)[n^{p^i(p-1)} + n^{p^i(p-2)} + \cdots + n^{p^i} + 1],$$

together with the fact that each of the p terms between the square brackets is congruent to 1 modulo p, implies that the result is also true for $i + 1$.

Hence $\left\lfloor \dfrac{n^{p^i}}{p^i} \right\rfloor = \dfrac{n^{p^i} - 1}{p^i}$, an odd integer for each $i \geqslant 1$.

Finally, we consider the case $n = 2$. We observe that $3 \cdot 4^i$ is a divisor of $2^{3 \cdot 4^i} - 4^i$ for every $i \geqslant 1$: Trivially, 4^i is a divisor of $2^{3\cdot 4^i} - 4^i$, since $3 \cdot 4^i > 2i$. Furthermore, since $2^{3\cdot 4^i}$ and 4^i are both congruent to 1 modulo 3, we have $3 \mid 2^{3\cdot 4^i} - 4^i$. Hence, $\left\lfloor \dfrac{2^{3\cdot 4^i}}{3 \cdot 4^i} \right\rfloor = \dfrac{2^{3\cdot 4^i} - 4^i}{3 \cdot 4^i} = \dfrac{2^{3\cdot 4^i - 2i} - 1}{3}$, which is odd for every $i \geqslant 1$.

Comment. The case n even and $n > 2$ can also be solved by recursively defining the sequence $(k_i)_{i\geqslant 1}$ by $k_1 = 1$ and $k_{i+1} = n^{k_i} - 1$ for $i \geqslant 1$. Then (k_i) is strictly increasing and it follows (by induction on i) that $k_i \mid n^{k_i} - 1$ for all $i \geqslant 1$, so the k_i are as desired.

The case $n = 2$ can also be solved as follows: Let $i \geqslant 2$. By BERTRAND's postulate, there exists a prime number p such that $2^{2^i - 1} < p \cdot 2^i < 2^{2^i}$. This gives

$$p \cdot 2^i < 2^{2^i} < 2p \cdot 2^i. \tag{1}$$

Also, we have that $p \cdot 2^i$ is a divisor of $2^{p\cdot 2^i} - 2^{2^i}$, hence, using (1), we get that

$$\left\lfloor \frac{2^{p\cdot 2^i}}{p \cdot 2^i} \right\rfloor = \frac{2^{p\cdot 2^i} - 2^{2^i} + p \cdot 2^i}{p \cdot 2^i} = \frac{2^{p\cdot 2^i - i} - 2^{2^i - i} + p}{p},$$

which is an odd integer.

Solution 3. Treat the (trivial) case when n is odd as in Solution 1.

Let n be even, and let p be a prime divisor of $n+1$. Define the sequence $(a_i)_{i\geqslant 1}$ by

$$a_i = \min\bigl\{a \in \mathbb{Z}_{>0} \colon 2^i \text{ divides } ap + 1\bigr\}.$$

Recall that there exists a with $1 \leqslant a < 2^i$ such that $ap \equiv -1 \pmod{2^i}$, so each a_i satisfies $1 \leqslant a_i < 2^i$. This implies that $a_i p + 1 < p \cdot 2^i$. Also, $a_i \to \infty$ as $i \to \infty$, whence there are infinitely many i such that $a_i < a_{i+1}$. From now on, we restrict ourselves only to these i.

Notice that p is a divisor of $n^p + 1$, which, in turn, divides $n^{p\cdot 2^i} - 1$. It follows that $p \cdot 2^i$ is a divisor of $n^{p\cdot 2^i} - (a_i p + 1)$, and we consequently see that the integer $\left\lfloor \dfrac{n^{p\cdot 2^i}}{p \cdot 2^i} \right\rfloor = \dfrac{n^{p\cdot 2^i} - (a_i p + 1)}{p \cdot 2^i}$ is odd, since 2^{i+1} divides $n^{p\cdot 2^i}$, but not $a_i p + 1$.

N5. Find all triples (p, x, y) consisting of a prime number p and two positive integers x and y such that $x^{p-1} + y$ and $x + y^{p-1}$ are both powers of p.

(Belgium)

Answer. $(p, x, y) \in \{(3, 2, 5), (3, 5, 2)\} \cup \{(2, n, 2^k - n) \mid 0 < n < 2^k\}$.

Solution 1. For $p = 2$, clearly all pairs of two positive integers x and y whose sum is a power of 2 satisfy the condition. Thus we assume in the following that $p > 2$, and we let a and b be positive integers such that $x^{p-1} + y = p^a$ and $x + y^{p-1} = p^b$. Assume further, without loss of generality, that $x \leqslant y$, so that $p^a = x^{p-1} + y \leqslant x + y^{p-1} = p^b$, which means that $a \leqslant b$ (and thus $p^a \mid p^b$).

Now we have

$$p^b = y^{p-1} + x = (p^a - x^{p-1})^{p-1} + x.$$

We take this equation modulo p^a and take into account that $p - 1$ is even, which gives us

$$0 \equiv x^{(p-1)^2} + x \pmod{p^a}.$$

If $p \mid x$, then $p^a \mid x$, since $x^{(p-1)^2-1} + 1$ is not divisible by p in this case. However, this is impossible, since $x \leqslant x^{p-1} < p^a$. Thus we know that $p \nmid x$, which means that

$$p^a \mid x^{(p-1)^2-1} + 1 = x^{p(p-2)} + 1.$$

By Fermat's little theorem, $x^{(p-1)^2} \equiv 1 \pmod{p}$, thus p divides $x+1$. Let p^r be the highest power of p that divides $x + 1$. By the binomial theorem, we have

$$x^{p(p-2)} = \sum_{k=0}^{p(p-2)} \binom{p(p-2)}{k} (-1)^{p(p-2)-k}(x + 1)^k.$$

Except for the terms corresponding to $k = 0$, $k = 1$ and $k = 2$, all terms in the sum are clearly divisible by p^{3r} and thus by p^{r+2}. The remaining terms are

$$-\frac{p(p-2)(p^2-2p-1)}{2}(x+1)^2,$$

which is divisible by p^{2r+1} and thus also by p^{r+2},

$$p(p-2)(x+1),$$

which is divisible by p^{r+1}, but not p^{r+2} by our choice of r, and the final term -1 corresponding to $k = 0$. It follows that the highest power of p that divides $x^{p(p-2)} + 1$ is p^{r+1}.

On the other hand, we already know that p^a divides $x^{p(p-2)} + 1$, which means that $a \leqslant r + 1$. Moreover,

$$p^r \leqslant x + 1 \leqslant x^{p-1} + y = p^a.$$

Hence we either have $a = r$ or $a = r + 1$.

If $a = r$, then $x = y = 1$ needs to hold in the inequality above, which is impossible for $p > 2$. Thus $a = r + 1$. Now since $p^r \leqslant x + 1$, we get

$$x = \frac{x^2 + x}{x + 1} \leqslant \frac{x^{p-1} + y}{x + 1} = \frac{p^a}{x + 1} \leqslant \frac{p^a}{p^r} = p,$$

so we must have $x = p - 1$ for p to divide $x + 1$.

It follows that $r = 1$ and $a = 2$. If $p \geqslant 5$, we obtain

$$p^a = x^{p-1} + y > (p - 1)^4 = (p^2 - 2p + 1)^2 > (3p)^2 > p^2 = p^a,$$

a contradiction. So the only case that remains is $p = 3$, and indeed $x = 2$ and $y = p^a - x^{p-1} = 5$ satisfy the conditions.

Comment 1. In this solution, we are implicitly using a special case of the following lemma known as "lifting the exponent":

Lemma. Let n be a positive integer, let p be an odd prime, and let $v_p(m)$ denote the exponent of the highest power of p that divides m.

If x and y are integers not divisible by p such that $p \mid x-y$, then we have

$$v_p(x^n - y^n) = v_p(x-y) + v_p(n).$$

Likewise, if x and y are integers not divisible by p such that $p \mid x+y$, then we have

$$v_p(x^n + y^n) = v_p(x+y) + v_p(n).$$

Comment 2. There exist various ways of solving the problem involving the "lifting the exponent" lemma. Let us sketch another one.

The cases $x = y$ and $p \mid x$ are ruled out easily, so we assume that $p > 2$, $x < y$, and $p \nmid x$. In this case we also have $p^a < p^b$ and $p \mid x+1$.

Now one has

$$y^p - x^p \equiv y(y^{p-1}+x) - x(x^{p-1}+y) \equiv 0 \pmod{p^a},$$

so by the lemma mentioned above one has $p^{a-1} \mid y-x$ and hence $y = x + tp^{a-1}$ for some positive integer t. Thus one gets

$$x(x^{p-2}+1) = x^{p-1} + x = (x^{p-1}+y) - (y-x) = p^{a-1}(p-t).$$

The factors on the left-hand side are coprime. So if $p \mid x$, then $x^{p-2}+1 \mid p-t$, which is impossible since $x < x^{p-2}+1$. Therefore, $p \nmid x$, and thus $x \mid p-t$. Since $p \mid x+1$, the only remaining case is $x = p-1$, $t = 1$, and $y = p^{a-1}+p-1$. Now the solution can be completed in the same way as before.

Solution 2. Again, we can focus on the case that $p > 2$. If $p \mid x$, then also $p \mid y$. In this case, let p^k and p^ℓ be the highest powers of p that divide x and y respectively, and assume without loss of generality that $k \leqslant \ell$. Then p^k divides $x + y^{p-1}$ while p^{k+1} does not, but $p^k < x + y^{p-1}$, which yields a contradiction. So x and y are not divisible by p. Fermat's little theorem yields $0 \equiv x^{p-1} + y \equiv 1 + y \pmod{p}$, so $y \equiv -1 \pmod{p}$ and for the same reason $x \equiv -1 \pmod{p}$.

In particular, $x, y \geqslant p-1$ and thus $x^{p-1} + y \geqslant 2(p-1) > p$, so $x^{p-1}+y$ and $y^{p-1}+x$ are both at least equal to p^2. Now we have

$$x^{p-1} \equiv -y \pmod{p^2} \qquad \text{and} \qquad y^{p-1} \equiv -x \pmod{p^2}.$$

These two congruences, together with the Euler–Fermat theorem, give us

$$1 \equiv x^{p(p-1)} \equiv (-y)^p \equiv -y^p \equiv xy \pmod{p^2}.$$

Since $x \equiv y \equiv -1 \pmod{p}$, $x-y$ is divisible by p, so $(x-y)^2$ is divisible by p^2. This means that

$$(x+y)^2 = (x-y)^2 + 4xy \equiv 4 \pmod{p^2},$$

so p^2 divides $(x+y-2)(x+y+2)$. We already know that $x+y \equiv -2 \pmod{p}$, so $x+y-2 \equiv -4 \not\equiv 0 \pmod{p}$. This means that p^2 divides $x+y+2$.

Using the same notation as in the first solution, we subtract the two original equations to obtain

$$p^b - p^a = y^{p-1} - x^{p-1} + x - y = (y-x)(y^{p-2} + y^{p-3}x + \cdots + x^{p-2} - 1). \tag{1}$$

The second factor is symmetric in x and y, so it can be written as a polynomial of the elementary symmetric polynomials $x+y$ and xy with integer coefficients. In particular, its value modulo

p^2 is characterised by the two congruences $xy \equiv 1 \pmod{p^2}$ and $x + y \equiv -2 \pmod{p^2}$. Since both congruences are satisfied when $x = y = -1$, we must have

$$y^{p-2} + y^{p-3}x + \cdots + x^{p-2} - 1 \equiv (-1)^{p-2} + (-1)^{p-3}(-1) + \cdots + (-1)^{p-2} - 1 \pmod{p^2},$$

which simplifies to $y^{p-2} + y^{p-3}x + \cdots + x^{p-2} - 1 \equiv -p \pmod{p^2}$. Thus the second factor in (1) is divisible by p, but not p^2.

This means that p^{a-1} has to divide the other factor $y - x$. It follows that

$$0 \equiv x^{p-1} + y \equiv x^{p-1} + x \equiv x(x+1)(x^{p-3} - x^{p-4} + \cdots + 1) \pmod{p^{a-1}}.$$

Since $x \equiv -1 \pmod{p}$, the last factor is $x^{p-3} - x^{p-4} + \cdots + 1 \equiv p - 2 \pmod{p}$ and in particular not divisible by p. We infer that $p^{a-1} \mid x + 1$ and continue as in the first solution.

Comment. Instead of reasoning by means of elementary symmetric polynomials, it is possible to provide a more direct argument as well. For odd r, $(x + 1)^2$ divides $(x^r + 1)^2$, and since p divides $x + 1$, we deduce that p^2 divides $(x^r + 1)^2$. Together with the fact that $xy \equiv 1 \pmod{p^2}$, we obtain

$$0 \equiv y^r(x^r + 1)^2 \equiv x^{2r}y^r + 2x^ry^r + y^r \equiv x^r + 2 + y^r \pmod{p^2}.$$

We apply this congruence with $r = p - 2 - 2k$ (where $0 \leqslant k < (p-2)/2$) to find that

$$x^ky^{p-2-k} + x^{p-2-k}y^k \equiv (xy)^k(x^{p-2-2k} + y^{p-2-2k}) \equiv 1^k \cdot (-2) \equiv -2 \pmod{p^2}.$$

Summing over all k yields

$$y^{p-2} + y^{p-3}x + \cdots + x^{p-2} - 1 \equiv \tfrac{p-1}{2} \cdot (-2) - 1 \equiv -p \pmod{p^2}$$

once again.

N6. Let $a_1 < a_2 < \cdots < a_n$ be pairwise coprime positive integers with a_1 being prime and $a_1 \geqslant n+2$. On the segment $I = [0, a_1 a_2 \cdots a_n]$ of the real line, mark all integers that are divisible by at least one of the numbers $a_1, \ldots, a_n$. These points split I into a number of smaller segments. Prove that the sum of the squares of the lengths of these segments is divisible by a_1.

(Serbia)

Solution 1. Let $A = a_1 \cdots a_n$. Throughout the solution, all intervals will be nonempty and have integer end-points. For any interval X, the length of X will be denoted by $|X|$.

Define the following two families of intervals:

$$\mathcal{S} = \big\{[x,y] \colon x < y \text{ are consecutive marked points}\big\}$$
$$\mathcal{T} = \big\{[x,y] \colon x < y \text{ are integers, } 0 \leqslant x \leqslant A-1, \text{ and no point is marked in } (x,y)\big\}$$

We are interested in computing $\sum\limits_{X \in \mathcal{S}} |X|^2$ modulo a_1.

Note that the number A is marked, so in the definition of $\mathcal{T}$ the condition $y \leqslant A$ is enforced without explicitly prescribing it.

Assign weights to the intervals in $\mathcal{T}$, depending only on their lengths. The weight of an arbitrary interval $Y \in \mathcal{T}$ will be $w\big(|Y|\big)$, where

$$w(k) = \begin{cases} 1 & \text{if } k = 1, \\ 2 & \text{if } k \geqslant 2\,. \end{cases}$$

Consider an arbitrary interval $X \in \mathcal{S}$ and its sub-intervals $Y \in \mathcal{T}$. Clearly, X has one sub-interval of length $|X|$, two sub-intervals of length $|X|-1$ and so on; in general X has $|X|-d+1$ sub-intervals of length d for every $d = 1, 2, \ldots, |X|$. The sum of the weights of the sub-intervals of X is

$$\sum_{Y \in \mathcal{T},\, Y \subseteq X} w\big(|Y|\big) = \sum_{d=1}^{|X|} (|X|-d+1) \cdot w(d) = |X| \cdot 1 + \big((|X|-1) + (|X|-2) + \cdots + 1\big) \cdot 2 = |X|^2.$$

Since the intervals in $\mathcal{S}$ are non-overlapping, every interval $Y \in \mathcal{T}$ is a sub-interval of a single interval $X \in \mathcal{S}$. Therefore,

$$\sum_{X \in \mathcal{S}} |X|^2 = \sum_{X \in \mathcal{S}} \left(\sum_{Y \in \mathcal{T},\, Y \subseteq X} w\big(|Y|\big) \right) = \sum_{Y \in \mathcal{T}} w\big(|Y|\big). \tag{1}$$

For every $d = 1, 2, \ldots, a_1$, we count how many intervals in $\mathcal{T}$ are of length d. Notice that the multiples of a_1 are all marked, so the lengths of the intervals in $\mathcal{S}$ and $\mathcal{T}$ cannot exceed a_1. Let x be an arbitrary integer with $0 \leqslant x \leqslant A-1$ and consider the interval $[x, x+d]$. Let r_1, $\ldots$, r_n be the remainders of x modulo a_1, $\ldots$, a_n, respectively. Since $a_1, \ldots, a_n$ are pairwise coprime, the number x is uniquely identified by the sequence $(r_1, \ldots, r_n)$, due to the Chinese remainder theorem.

For every $i = 1, \ldots, n$, the property that the interval $(x, x+d)$ does not contain any multiple of a_i is equivalent with $r_i + d \leqslant a_i$, i.e. $r_i \in \{0, 1, \ldots, a_i - d\}$, so there are $a_i - d + 1$ choices for the number r_i for each i. Therefore, the number of the remainder sequences $(r_1, \ldots, r_n)$ that satisfy $[x, x+d] \in \mathcal{T}$ is precisely $(a_1 + 1 - d) \cdots (a_n + 1 - d)$. Denote this product by $f(d)$.

Now we can group the last sum in (1) by length of the intervals. As we have seen, for every $d = 1, \ldots, a_1$ there are $f(d)$ intervals $Y \in \mathcal{T}$ with $|Y| = d$. Therefore, (1) can be continued as

$$\sum_{X \in \mathcal{S}} |X|^2 = \sum_{Y \in \mathcal{T}} w\big(|Y|\big) = \sum_{d=1}^{a_1} f(d) \cdot w(d) = 2 \sum_{d=1}^{a_1} f(d) - f(1). \tag{2}$$

Having the formula (2), the solution can be finished using the following well-known fact:
Lemma. If p is a prime, $F(x)$ is a polynomial with integer coefficients, and $\deg F \leqslant p-2$, then $\sum\limits_{x=1}^{p} F(x)$ is divisible by p.
Proof. Obviously, it is sufficient to prove the lemma for monomials of the form x^k with $k \leqslant p-2$. Apply induction on k. If $k = 0$ then $F = 1$, and the statement is trivial.

Let $1 \leqslant k \leqslant p-2$, and assume that the lemma is proved for all lower degrees. Then

$$\begin{aligned} 0 \equiv p^{k+1} &= \sum_{x=1}^{p} \big(x^{k+1} - (x-1)^{k+1}\big) = \sum_{x=1}^{p} \left(\sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k+1}{\ell} x^\ell \right) \\ &= (k+1) \sum_{x=1}^{p} x^k + \sum_{\ell=0}^{k-1} (-1)^{k-\ell} \binom{k+1}{\ell} \sum_{x=1}^{p} x^\ell \equiv (k+1) \sum_{x=1}^{p} x^k \pmod{p}. \end{aligned}$$

Since $0 < k+1 < p$, this proves $\sum\limits_{x=1}^{p} x^k \equiv 0 \pmod{p}$. □

In (2), by applying the lemma to the polynomial f and the prime a_1, we obtain that $\sum\limits_{d=1}^{a_1} f(d)$ is divisible by a_1. The term $f(1) = a_1 \cdots a_n$ is also divisible by a_1; these two facts together prove that $\sum\limits_{X \in \mathcal{S}} |X|^2$ is divisible by a_1.

Comment 1. With suitable sets of weights, the same method can be used to sum up other expressions on the lengths of the segments. For example, $w(1) = 1$ and $w(k) = 6(k-1)$ for $k \geqslant 2$ can be used to compute $\sum\limits_{X \in \mathcal{S}} |X|^3$ and to prove that this sum is divisible by a_1 if a_1 is a prime with $a_1 \geqslant n+3$. See also Comment 2 after the second solution.

Solution 2. The conventions from the first paragraph of the first solution are still in force. We shall prove the following more general statement:

> ($\boxplus$) *Let p denote a prime number, let $p = a_1 < a_2 < \cdots < a_n$ be n pairwise coprime positive integers, and let d be an integer with $1 \leqslant d \leqslant p-n$. Mark all integers that are divisible by at least one of the numbers $a_1, \ldots, a_n$ on the interval $I = [0, a_1 a_2 \cdots a_n]$ of the real line. These points split I into a number of smaller segments, say of lengths $b_1, \ldots, b_k$. Then the sum $\sum\limits_{i=1}^{k} \binom{b_i}{d}$ is divisible by p.*

Applying ($\boxplus$) to $d = 1$ and $d = 2$ and using the equation $x^2 = 2\binom{x}{2} + \binom{x}{1}$, one easily gets the statement of the problem.

To prove ($\boxplus$) itself, we argue by induction on n. The base case $n = 1$ follows from the known fact that the binomial coefficient $\binom{p}{d}$ is divisible by p whenever $1 \leqslant d \leqslant p-1$.

Let us now assume that $n \geqslant 2$, and that the statement is known whenever $n-1$ rather than n coprime integers are given together with some integer $d \in [1, p-n+1]$. Suppose that the numbers $p = a_1 < a_2 < \cdots < a_n$ and d are as above. Write $A' = \prod_{i=1}^{n-1} a_i$ and $A = A' a_n$. Mark the points on the real axis divisible by one of the numbers $a_1, \ldots, a_{n-1}$ *green* and those divisible by a_n *red*. The green points divide $[0, A']$ into certain sub-intervals, say J_1, J_2, $\ldots$, and J_ℓ.

To translate intervals we use the notation $[a, b] + m = [a + m, b + m]$ whenever $a, b, m \in \mathbb{Z}$.

For each $i \in \{1, 2, \ldots, \ell\}$ let $\mathcal{F}_i$ be the family of intervals into which the red points partition the intervals J_i, $J_i + A'$, $\ldots$, and $J_i + (a_n - 1)A'$. We are to prove that

$$\sum_{i=1}^{\ell} \sum_{X \in \mathcal{F}_i} \binom{|X|}{d}$$

is divisible by p.

Let us fix any index i with $1 \leqslant i \leqslant \ell$ for a while. Since the numbers A' and a_n are coprime by hypothesis, the numbers $0, A', \ldots, (a_n - 1)A'$ form a complete system of residues modulo a_n. Moreover, we have $|J_i| \leqslant p < a_n$, as in particular all multiples of p are green. So each of the intervals J_i, $J_i + A'$, $\ldots$, and $J_i + (a_n - 1)A'$ contains at most one red point. More precisely, for each $j \in \{1, \ldots, |J_i| - 1\}$ there is exactly one amongst those intervals containing a red point splitting it into an interval of length j followed by an interval of length $|J_i| - j$, while the remaining $a_n - |J_i| + 1$ such intervals have no red points in their interiors. For these reasons

$$\begin{aligned}
\sum_{X \in \mathcal{F}_i} \binom{|X|}{d} &= 2\left(\binom{1}{d} + \cdots + \binom{|J_i| - 1}{d}\right) + (a_n - |J_i| + 1)\binom{|J_i|}{d} \\
&= 2\binom{|J_i|}{d+1} + (a_n - d + 1)\binom{|J_i|}{d} - (d+1)\binom{|J_i|}{d+1} \\
&= (1 - d)\binom{|J_i|}{d+1} + (a_n - d + 1)\binom{|J_i|}{d}.
\end{aligned}$$

So it remains to prove that

$$(1 - d)\sum_{i=1}^{\ell} \binom{|J_i|}{d+1} + (a_n - d + 1)\sum_{i=1}^{\ell} \binom{|J_i|}{d}$$

is divisible by p. By the induction hypothesis, however, it is even true that both summands are divisible by p, for $1 \leqslant d < d + 1 \leqslant p - (n - 1)$. This completes the proof of ($\boxplus$) and hence the solution of the problem.

Comment 2. The statement ($\boxplus$) can also be proved by the method of the first solution, using the weights $w(x) = \binom{x-2}{d-2}$.

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N7. Let $c \geqslant 1$ be an integer. Define a sequence of positive integers by $a_1 = c$ and

$$a_{n+1} = a_n^3 - 4c \cdot a_n^2 + 5c^2 \cdot a_n + c$$

for all $n \geqslant 1$. Prove that for each integer $n \geqslant 2$ there exists a prime number p dividing a_n but none of the numbers $a_1, \ldots, a_{n-1}$.

(Austria)

Solution. Let us define $x_0 = 0$ and $x_n = a_n/c$ for all integers $n \geqslant 1$. It is easy to see that the sequence (x_n) thus obtained obeys the recursive law

$$x_{n+1} = c^2(x_n^3 - 4x_n^2 + 5x_n) + 1 \tag{1}$$

for all integers $n \geqslant 0$. In particular, all of its terms are positive integers; notice that $x_1 = 1$ and $x_2 = 2c^2 + 1$. Since

$$x_{n+1} = c^2 x_n (x_n - 2)^2 + c^2 x_n + 1 > x_n \tag{2}$$

holds for all integers $n \geqslant 0$, it is also strictly increasing. Since x_{n+1} is by (1) coprime to c for any $n \geqslant 0$, it suffices to prove that for each $n \geqslant 2$ there exists a prime number p dividing x_n but none of the numbers $x_1, \ldots, x_{n-1}$. Let us begin by establishing three preliminary claims.

Claim 1. If $i \equiv j \pmod{m}$ holds for some integers $i, j \geqslant 0$ and $m \geqslant 1$, then $x_i \equiv x_j \pmod{x_m}$ holds as well.

Proof. Evidently, it suffices to show $x_{i+m} \equiv x_i \pmod{x_m}$ for all integers $i \geqslant 0$ and $m \geqslant 1$. For this purpose we may argue for fixed m by induction on i using $x_0 = 0$ in the base case $i = 0$. Now, if we have $x_{i+m} \equiv x_i \pmod{x_m}$ for some integer i, then the recursive equation (1) yields

$$x_{i+m+1} \equiv c^2(x_{i+m}^3 - 4x_{i+m}^2 + 5x_{i+m}) + 1 \equiv c^2(x_i^3 - 4x_i^2 + 5x_i) + 1 \equiv x_{i+1} \pmod{x_m},$$

which completes the induction. □

Claim 2. If the integers $i, j \geqslant 2$ and $m \geqslant 1$ satisfy $i \equiv j \pmod{m}$, then $x_i \equiv x_j \pmod{x_m^2}$ holds as well.

Proof. Again it suffices to prove $x_{i+m} \equiv x_i \pmod{x_m^2}$ for all integers $i \geqslant 2$ and $m \geqslant 1$. As above, we proceed for fixed m by induction on i. The induction step is again easy using (1), but this time the base case $i = 2$ requires some calculation. Set $L = 5c^2$. By (1) we have $x_{m+1} \equiv Lx_m + 1 \pmod{x_m^2}$, and hence

$$\begin{aligned} x_{m+1}^3 - 4x_{m+1}^2 + 5x_{m+1} &\equiv (Lx_m + 1)^3 - 4(Lx_m + 1)^2 + 5(Lx_m + 1) \\ &\equiv (3Lx_m + 1) - 4(2Lx_m + 1) + 5(Lx_m + 1) \equiv 2 \pmod{x_m^2}, \end{aligned}$$

which in turn gives indeed $x_{m+2} \equiv 2c^2 + 1 \equiv x_2 \pmod{x_m^2}$. □

Claim 3. For each integer $n \geqslant 2$, we have $x_n > x_1 \cdot x_2 \cdots x_{n-2}$.

Proof. The cases $n = 2$ and $n = 3$ are clear. Arguing inductively, we assume now that the claim holds for some $n \geqslant 3$. Recall that $x_2 \geqslant 3$, so by monotonicity and (2) we get $x_n \geqslant x_3 \geqslant x_2(x_2 - 2)^2 + x_2 + 1 \geqslant 7$. It follows that

$$x_{n+1} > x_n^3 - 4x_n^2 + 5x_n > 7x_n^2 - 4x_n^2 > x_n^2 > x_n x_{n-1},$$

which by the induction hypothesis yields $x_{n+1} > x_1 \cdot x_2 \cdots x_{n-1}$, as desired. □

Now we direct our attention to the problem itself: let any integer $n \geqslant 2$ be given. By Claim 3 there exists a prime number p appearing with a higher exponent in the prime factorisation of x_n than in the prime factorisation of $x_1 \cdots x_{n-2}$. In particular, $p \mid x_n$, and it suffices to prove that p divides none of $x_1, \ldots, x_{n-1}$.

Otherwise let $k \in \{1, \ldots, n-1\}$ be minimal such that p divides x_k. Since x_{n-1} and x_n are coprime by (1) and $x_1 = 1$, we actually have $2 \leqslant k \leqslant n-2$. Write $n = qk + r$ with some integers $q \geqslant 0$ and $0 \leqslant r < k$. By Claim 1 we have $x_n \equiv x_r \pmod{x_k}$, whence $p \mid x_r$. Due to the minimality of k this entails $r = 0$, i.e. $k \mid n$.

Thus from Claim 2 we infer

$$x_n \equiv x_k \pmod{x_k^2}.$$

Now let $\alpha \geqslant 1$ be maximal with the property $p^\alpha \mid x_k$. Then x_k^2 is divisible by $p^{\alpha+1}$ and by our choice of p so is x_n. So by the previous congruence x_k is a multiple of $p^{\alpha+1}$ as well, contrary to our choice of α. This is the final contradiction concluding the solution.

N8. For every real number x, let $\|x\|$ denote the distance between x and the nearest integer. Prove that for every pair (a, b) of positive integers there exist an odd prime p and a positive integer k satisfying

$$\left\|\frac{a}{p^k}\right\| + \left\|\frac{b}{p^k}\right\| + \left\|\frac{a+b}{p^k}\right\| = 1. \tag{1}$$

(Hungary)

Solution. Notice first that $\left\lfloor x+\frac{1}{2}\right\rfloor$ is an integer nearest to x, so $\|x\| = \left|\left\lfloor x+\frac{1}{2}\right\rfloor - x\right|$. Thus we have

$$\left\lfloor x+\frac{1}{2}\right\rfloor = x \pm \|x\|. \tag{2}$$

For every rational number r and every prime number p, denote by $v_p(r)$ the exponent of p in the prime factorisation of r. Recall the notation $(2n-1)!!$ for the product of all odd positive integers not exceeding $2n-1$, i.e., $(2n-1)!! = 1 \cdot 3 \cdots (2n-1)$.

Lemma. For every positive integer n and every odd prime p, we have

$$v_p\big((2n-1)!!\big) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} + \frac{1}{2} \right\rfloor.$$

Proof. For every positive integer k, let us count the multiples of p^k among the factors $1, 3, \ldots, 2n-1$. If ℓ is an arbitrary integer, the number $(2\ell-1)p^k$ is listed above if and only if

$$0 < (2\ell-1)p^k \leqslant 2n \quad \Longleftrightarrow \quad \frac{1}{2} < \ell \leqslant \frac{n}{p^k} + \frac{1}{2} \quad \Longleftrightarrow \quad 1 \leqslant \ell \leqslant \left\lfloor \frac{n}{p^k} + \frac{1}{2} \right\rfloor.$$

Hence, the number of multiples of p^k among the factors is precisely $m_k = \left\lfloor \frac{n}{p^k} + \frac{1}{2} \right\rfloor$. Thus we obtain

$$v_p\big((2n-1)!!\big) = \sum_{i=1}^{n} v_p(2i-1) = \sum_{i=1}^{n} \sum_{k=1}^{v_p(2i-1)} 1 = \sum_{k=1}^{\infty} \sum_{\ell=1}^{m_k} 1 = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} + \frac{1}{2} \right\rfloor. \qquad \square$$

In order to prove the problem statement, consider the rational number

$$N = \frac{(2a+2b-1)!!}{(2a-1)!! \cdot (2b-1)!!} = \frac{(2a+1)(2a+3)\cdots(2a+2b-1)}{1 \cdot 3 \cdots (2b-1)}.$$

Obviously, $N > 1$, so there exists a prime p with $v_p(N) > 0$. Since N is a fraction of two odd numbers, p is odd.

By our lemma,

$$0 < v_p(N) = \sum_{k=1}^{\infty} \left(\left\lfloor \frac{a+b}{p^k} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{a}{p^k} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{b}{p^k} + \frac{1}{2} \right\rfloor \right).$$

Therefore, there exists some positive integer k such that the integer number

$$d_k = \left\lfloor \frac{a+b}{p^k} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{a}{p^k} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{b}{p^k} + \frac{1}{2} \right\rfloor$$

is positive, so $d_k \geqslant 1$. By (2) we have

$$1 \leqslant d_k = \frac{a+b}{p^k} - \frac{a}{p^k} - \frac{b}{p^k} \pm \left\|\frac{a+b}{p^k}\right\| \pm \left\|\frac{a}{p^k}\right\| \pm \left\|\frac{b}{p^k}\right\| = \pm \left\|\frac{a+b}{p^k}\right\| \pm \left\|\frac{a}{p^k}\right\| \pm \left\|\frac{b}{p^k}\right\|. \tag{3}$$

Since $\|x\| < \frac{1}{2}$ for every rational x with odd denominator, the relation (3) can only be satisfied if all three signs on the right-hand side are positive and $d_k = 1$. Thus we get

$$\left\|\frac{a}{p^k}\right\| + \left\|\frac{b}{p^k}\right\| + \left\|\frac{a+b}{p^k}\right\| = d_k = 1,$$

as required.

Comment 1. There are various choices for the number N in the solution. Here we sketch such a version.

Let x and y be two rational numbers with odd denominators. It is easy to see that the condition $\|x\| + \|y\| + \|x+y\| = 1$ is satisfied if and only if

$$\text{either}\quad \{x\} < \tfrac{1}{2}, \quad \{y\} < \tfrac{1}{2}, \quad \{x+y\} > \tfrac{1}{2}, \qquad \text{or}\quad \{x\} > \tfrac{1}{2}, \quad \{y\} > \tfrac{1}{2}, \quad \{x+y\} < \tfrac{1}{2},$$

where $\{x\}$ denotes the fractional part of x.

In the context of our problem, the first condition seems easier to deal with. Also, one may notice that

$$\{x\} < \tfrac{1}{2} \iff \varkappa(x) = 0 \qquad \text{and} \qquad \{x\} \geqslant \tfrac{1}{2} \iff \varkappa(x) = 1, \tag{4}$$

where

$$\varkappa(x) = \lfloor 2x \rfloor - 2\lfloor x \rfloor.$$

Now it is natural to consider the number

$$M = \frac{\dbinom{2a+2b}{a+b}}{\dbinom{2a}{a}\dbinom{2b}{b}},$$

since

$$v_p(M) = \sum_{k=1}^{\infty} \left(\varkappa\left(\frac{2(a+b)}{p^k}\right) - \varkappa\left(\frac{2a}{p^k}\right) - \varkappa\left(\frac{2b}{p^k}\right) \right).$$

One may see that $M > 1$, and that $v_2(M) \leqslant 0$. Thus, there exist an odd prime p and a positive integer k with

$$\varkappa\left(\frac{2(a+b)}{p^k}\right) - \varkappa\left(\frac{2a}{p^k}\right) - \varkappa\left(\frac{2b}{p^k}\right) > 0.$$

In view of (4), the last inequality yields

$$\left\{\frac{a}{p^k}\right\} < \frac{1}{2}, \quad \left\{\frac{b}{p^k}\right\} < \frac{1}{2}, \quad \text{and} \quad \left\{\frac{a+b}{p^k}\right\} > \frac{1}{2}, \tag{5}$$

which is what we wanted to obtain.

Comment 2. Once one tries to prove the existence of suitable p and k satisfying (5), it seems somehow natural to suppose that $a \leqslant b$ and to add the restriction $p^k > a$. In this case the inequalities (5) can be rewritten as

$$2a < p^k, \quad 2mp^k < 2b < (2m+1)p^k, \quad \text{and} \quad (2m+1)p^k < 2(a+b) < (2m+2)p^k$$

for some positive integer m. This means exactly that one of the numbers $2a+1, 2a+3, \ldots, 2a+2b-1$ is divisible by some number of the form p^k which is greater than $2a$.

Using more advanced techniques, one can show that such a number p^k exists even with $k = 1$. This was shown in 2004 by Laishram and Shorey; the methods used for this proof are elementary but still quite involved. In fact, their result generalises a theorem by Sylvester which states that for every pair of integers (n, k) with $n \geqslant k \geqslant 1$, the product $(n+1)(n+2)\cdots(n+k)$ is divisible by some prime $p > k$. We would like to mention here that Sylvester's theorem itself does not seem to suffice for solving the problem.